VECTOR ANALYSIS

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PREFACE

MODERN vector analysis is essentially an offshoot of the work of Grassmann and of the quaternionic investigations of Hamilton, both rich sources of mathematical knowledge. Neither line of inquiry has received much development. Hamilton did indeed originate schools of workers following his methods, but the number of these who have really made the quaternion method their own and have converted it into a useful weapon for further research is indeed small.

As for Grassmann, his work initially remained quite unnoticed, and even to-day is rarely studied. Portions of it have penetrated into mathematical teaching and into the exact sciences, to the permanent advantage of the investigator. In my opinion, the methods and lines of inquiry opened up by Grassmann are far from exhausted. Many more of his ideas will yet become the property of the mathematician and of the mathematical investigator, and will be absorbed into general teaching. In the present work I have striven to correlate the customary conceptions of vector analysis with the fundamental ideas of Grassmann's treatment. I have, however, not always kept close to his terminology. It would, for example, be futile to attempt to discard the term "vector" even if it be admitted that "interval" (streeke) is

better. As far as the representation of scalar and vector products are concerned, I have not followed the system adopted by the "Mathematische Enzyklopädie," but rather that introduced by Gibbs. For the round and the square brackets, which are utilised for representing these operations in the "Mathematische Enzyklopädie," cannot be dispensed with for their normal interpretation in applications of the distributive law. Because of this dual interpretation of the brackets the formulæ become clumsy and difficult to check. In the treatment of tensors I owe much to Gibbs, for in this connection Grassmann's notation is too special.

This, the first volume, contains the vectorial analysis of three dimensions. In the second volume, that of four and more dimensions, playing an important part in the theory of relativity, will be treated. In that case the conception of the tensor is of first importance. It is one of those ideas to which Grassmann so prophetically referred at the conclusion of his addresses in 1862:—

"Ich weiss dass einst diese Ideen, wenn auch in veränderter Form, neu entstehen und mit der Zeitentwicklung in lebendige Wechselwirkung treten werden."

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VECTOR ANALYSIS

CHAPTER I

VECTORS AND VECTORIAL AREAS

§ 1. THE CONCEPTION OF A VECTOR

LET A and A′ be two given points in space. Passing outwards from both points in the same direction through the same rectilinear distance we obtain the two points B and B′ respectively. We then say that the "vector" which leads from A to B leads likewise from A′ to B′, or the vector that leads from A to B is equal to that leading from A′ to B′. A vector is thus an entity possessing direction and length and is completely characterised by these two properties. Two vectors are unequal when either their directions or their lengths, or both their directions and their lengths, are unequal. Two vectors having the same length and the same direction are equal.

We propose to represent a vector with small heavy (Clarendon) type, e.g. **a** or **b** or **p** or **x** in contrast to quantities representing positive or negative numbers which will be represented by small *italic* or Greek letters, or to points which will be indicated by large Roman letters. Moreover, the form of equation customary in algebra

$$\mathbf{a} = \mathbf{b}$$

will be used to express the idea that the two vectors do not differ from each other.

All quantities with which a direction may be associated, as for example a speed, an impulse, a force, an acceleration

and many other types of quantity which arise in the consideration of physical phenomena, may be represented geometrically by vectors as soon as we have determined what length of the unit of measurement is to be associated with the quantity in question. The directed quantity of n units of measurement is then represented geometrically by a vector in the same direction and n units of length.

§ 2. The Addition of Vectors

From two arbitrary vectors **a** and **b** a third vector **c** may be derived in the following manner. From an arbitrary point A draw the vector **a** terminating at a point B, and from B draw the vector **b** terminating at the point C; then **c** is the vector which leads from A to C. This derivation of **c** by the association of **a** and **b** we propose to represent by the plus sign + thus:

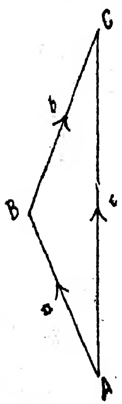


Fig. 1.

$$\mathbf{a} + \mathbf{b}$$

so that we say the two vectors **a** and **b** are added together.

That the same sign and the same word should here be used as occurs in algebra in a different connection will, as we shall see presently, be fully justified in that the laws for the addition of vectors are perfectly consistent with those for the addition of mere numbers. It is a great simplification to utilise the same sign; at the same time there is no possibility of confusion so long as our method of representing vectors (by Clarendon type) is different from our method of representing mere numbers (by *italic* letters).

That the vector **c** arises by the association of **a** and **b** in the manner described is conveyed by the equation:

$$(1) \qquad \mathbf{a} + \mathbf{b} = \mathbf{c}.$$

If the vector **b** is set off from the point A first terminating at the point B′ (fig. 2), then the vector **a** when set off from

B′ leads to the same point C as before and the result of the addition in this order is to lead to the same vector **c**. This fact is expressed by means of the equation :

(2) $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

It follows that just as in the addition of two numbers the order of summation may be reversed without affecting the sum, so the two constituent vectors in the above vectorial summation may be interchanged without affecting the result.

If the two constituent vectors are equal in length but opposite in direction then the point C coincides with the

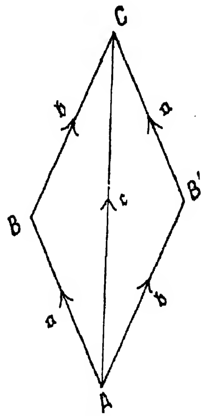


FIG. 2.

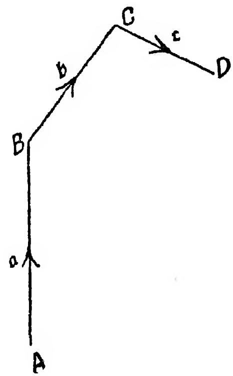


FIG. 3.

point A on adding and the vector **c** vanishes. As in the sum of two equal and opposite numbers, we may write this

$$\mathbf{a} + \mathbf{b} = 0,$$

or also. $$\mathbf{a} = -\mathbf{b}.$$

Under these circumstances we say the vectors are opposite.

Let A, B, C, D be four given points in space, and let **a**, **b**, **c** be the vectors leading from A to B, from B to C, and from C to D respectively (fig. 3). Adding **a** and **b** we derive the vector **a** + **b** and to this we add **c** ; hence we obtain the vector

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

leading from A to D. On the other hand, if the vectors $\mathbf{b}$ and $\mathbf{c}$ are added together so that we have $\mathbf{b} + \mathbf{c}$ leading from B to D and if to this we add $\mathbf{a}$ we obtain the vector

$$\mathbf{a} + (\mathbf{b} + \mathbf{c})$$

also leading from A to D. Accordingly we have

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}), \tag{3}$$

or, in other words, in adding

$$\mathbf{a} + \mathbf{b} + \mathbf{c}$$

the position of the brackets is immaterial. Now since in adding two vectors $\mathbf{a} + \mathbf{b}$ or $\mathbf{b} + \mathbf{c}$ the two vectors may be interchanged, it appears that in the summation

$$\mathbf{a} + \mathbf{b} + \mathbf{c}$$

each vector may be interchanged with its neighbour without affecting the final result and each of the three terms may be allowed to occupy any one of the three possible positions; in other words, just as in the addition of three numbers the order of summation does not influence the result. The six possible arrangements of the three terms correspond to six broken lines leading from A to D each with three sides formed from $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in some sequence.

Now suppose there are n vectors,

$$\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ . \ . \ . \ \mathbf{a}_n,$$

leading respectively from A_0 to A_1, etc., to A_n, then by adding $\mathbf{a}_1$ to $\mathbf{a}_2$, then $\mathbf{a}_3$ to the resultant and so on as far as $\mathbf{a}_n$, a vector is finally derived which leads from A_0 to A_n. But the same vector arises if groups of these constituent vectors are first associated; for example, if $\mathbf{a}_i$ to $\mathbf{a}_{i+k}$ are associated together to form the vector leading from A_{i-1} to A_{i+k} the points A_i to A_{i+k-1} would be omitted from the sequence $A_0 \ . \ . \ . \ A_n$. That in both cases the same resultant vector is derived can be expressed by means of the equation:

$$\begin{aligned} (4)\quad & \mathbf{a}_1 + \mathbf{a}_2 + \ . \ . \ . \ + \mathbf{a}_{i} + \ . \ . \ . \ + \mathbf{a}_{i+k} + \ . \ . \ . \ + \mathbf{a}_n \\ & = \mathbf{a}_1 + \mathbf{a}_2 + \ . \ . \ . \ + \mathbf{a}_{i-1} + (\mathbf{a}_i + \ . \ . \ . \ + \mathbf{a}_{i+k}) + \\ & \qquad \mathbf{a}_{i+k+1} + \mathbf{a}_n. \end{aligned}$$

In other words, the resultant vector

$$\mathbf{a}_0 + \mathbf{a}_1 + \ . \ . \ . \ + \mathbf{a}_n$$

is not affected by enclosing any sequence of terms in a bracket. From this it follows at once that the resultant vector is unaffected if any term $\mathbf{a}_i$ is interchanged with its neighbour $\mathbf{a}_{i+1}$. For if $\mathbf{a}_i + \mathbf{a}_{i+1}$ are bracketed together, within that bracket these two terms, as we have already seen, may be interchanged and then the brackets may be dropped. Geometrically this means that in the sequence of points A_0, A_1, . . . A_n the point A_i is replaced by another A_i' which is reached from A_{i-1} by the vector $\mathbf{a}_{i+1}$, the vector $\mathbf{a}_i$ then leading from A_i' to A_{i+1}. By successively interchanging two neighbouring terms in the expression

$$\mathbf{a}_1 + \mathbf{a}_2 + \ . \ . \ . \ + \mathbf{a}_n$$

any term may be transposed to any arbitrary position; as with the summation of any arbitrary number of ordinary numbers the resultant in this case is independent of the order of the terms in the sequence.

§ 3. Vectorial Equations

If the first point A_0 of the sequence A_0, A_1, A_2 . . . A_n coincides with the last point A_n the resultant vector vanishes. This may be expressed by means of the equation:

$$(1) \qquad \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \ . \ . \ . \ + \mathbf{a}_n = 0.$$

An equation of the form

$$(2) \qquad \mathbf{a}_1 + \mathbf{a}_2 + \ . \ . \ . \ + \mathbf{a}_r = \mathbf{b}_1 + \mathbf{b}_2 + \ . \ . \ . \ + \mathbf{b}_s$$

indicates that the vector formed by the addition of the vectors $\mathbf{a}_1$ . . . $\mathbf{a}_r$ is equal to the vector formed by the addition of the vectors $\mathbf{b}_1$ . . . $\mathbf{b}_s$, where it is not impossible that the resultant in both cases is zero. Such equations we term "vectorial equations." Any term may be transposed to the opposite side of the equation provided its sign be altered. For since the same vector may be added to both sides of the equation without invalidating it, it is merely necessary for example to add the vector $-\ \mathbf{b}_i$ to both

sides. On the right-hand side $\mathbf{b}_i$ and $-\mathbf{b}_i$ may then be brought together and bracketed, thus giving zero, so that they may be dropped out together. The result is such that $\mathbf{b}_i$ disappears entirely from the right-hand side while on the left-hand side it appears as $-\mathbf{b}_i$, written thus for short instead of $+(-\mathbf{b}_i)$.

If all the vectors except one in a vectorial equation are known, then the latter may be determined by bringing all the remaining vectors to the other side of the equation.

For example, it follows from

$$\mathbf{a} + \mathbf{x} = \mathbf{b}, \tag{3}$$

where $\mathbf{a}$ and $\mathbf{b}$ are known and $\mathbf{x}$ unknown, that

$$\mathbf{x} = \mathbf{b} - \mathbf{a} \tag{4}$$

by bringing $\mathbf{a}$ to the other side of the equation.

§ 4. Examples of the Addition of Vectors

The association of vectors plays a fundamental part in the consideration of all directed quantities. If, for example, a body moves in a plane with a velocity represented by a vector $\mathbf{a}$, and a second body moves relative to the first with a velocity $\mathbf{b}$, then the total velocity of the latter is $\mathbf{a} + \mathbf{b}$. If this total velocity is represented by $\mathbf{c}$ then its relative velocity is $\mathbf{c} - \mathbf{a}$. A ship travelling with a velocity $\mathbf{a}$ and acted on by a wind with velocity $\mathbf{c}$, experiences a relative wind $\mathbf{c} - \mathbf{a}$. The flag at the mast or the smoke at the funnel indicates the relative wind (fig. 4).

Two forces acting at a point combine according to the so-called parallelogram law, that is, their effect can be represented by that of a single force (the *resultant*) whose direction and magnitude is determined by the diagonal of the parallelogram whose sides represent the two forces. If the two vectors $\mathbf{a}$ and $\mathbf{b}$ represent the two forces then the resultant is $\mathbf{a} + \mathbf{b}$. If more than two forces operate at the point then they may be combined in this manner into a single force, which acting alone would produce the same

result as all the other forces acting together. If the latter are represented by $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ then the resultant is:

$$\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n.$$

In the special case where the system of forces is in equilibrium then the resultant vanishes, and accordingly:

$$\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n = 0. \tag{1}$$

Interpreted geometrically this means, as has been seen above,

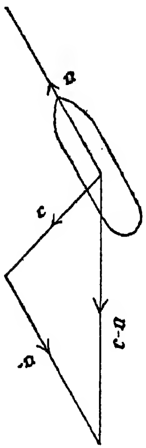


FIG. 4.

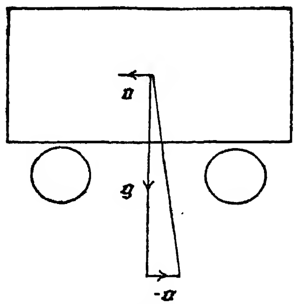


FIG. 5.

that the constituent forces when set off one at the end of the other must form a closed polygon.

Suppose a tramway car starts to move with constant acceleration so that the gain in speed per second is represented by the vector **a**; a body falling freely in the carriage during this accelerated motion experiences relative to the earth the same acceleration as every other freely falling body. Let the vector **g** represent the gain per second in speed acquired as a consequence of the force of gravity; relative to the carriage, however, it experiences an acceleration specified by $\mathbf{g} - \mathbf{a}$. A body falling from relative rest in the carriage moves relative to the latter in the direction of the vector

identical with that used in algebra. It may be extended to an arbitrary number of terms. For if the formula

(4) $a\mathbf{a}_1 + a\mathbf{a}_2 + \ldots + a\mathbf{a}_{n-1} = a(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_{n-1})$

be assumed true for $n - 1$ vectors, it may be immediately proved true for n vectors. Let, in fact,

$$\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_{n-1} = \mathbf{c},$$

then for the two vectors $\mathbf{c}$ and $\mathbf{a}_n$

$$a\mathbf{c} + a\mathbf{a}_n = a(\mathbf{c} + \mathbf{a}_n).$$

Now insert for $\mathbf{c}$ its value, then on the given assumption it follows that:

(5) $a\mathbf{a}_1 + a\mathbf{a}_2 + \ldots + a\mathbf{a}_n = a(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n).$

The equation may be geometrically interpreted in the following manner:—

Commencing with a point A_0, set off the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ one at the end of the other in succession; in this way a broken line $A_0A_1A_2 \ldots A_n$ is formed, where some of the $(n + 1)$ points A_0, etc., occupying any positions in space, may in particular cases coincide. The vector leading directly from A_0 to A_n is then:

$$\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n.$$

Multiplication by a positive number a implies that the whole figure is increased in scale a-fold. All the vectors

$\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n$ then become $a\mathbf{a}_1, a\mathbf{a}_2, \ldots a\mathbf{a}_n$

and $\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n$

becomes $a(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n)$.

At the same time it still remains true that the latter vector is composed of the sum of the others. With a negative value for it, in addition to a change of scale, the direction of all the vectors is reversed.

§ 6. Numerical Derivation of a Vector from a Set of Others

Every vector $\mathbf{p}$ having the same direction as another vector $\mathbf{a}$ or the opposite direction, can be derived from the

latter by multiplying by a positive or a negative number; thus:

(1) $$\mathbf{p} = x\mathbf{a}.$$

If two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are neither equal nor oppositely directed, and if a third vector $\mathbf{p}$ is derived from these two by means of the equation

(2) $$\mathbf{p} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2,$$

then the three vectors $\mathbf{p}$, $\mathbf{a}_1$ and $\mathbf{a}_2$ are all parallel to the same plane. For according to what has been already said, it follows that if the three vectors $-\mathbf{p}$, $x_1\mathbf{a}_1$, $x_2\mathbf{a}_2$ are set off in succession each from the extremity of the preceding, they will form a closed triangle. Every vector $\mathbf{q}$, parallel to the same plane, may be thrown into the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

by a suitable choice of x_1 and x_2.

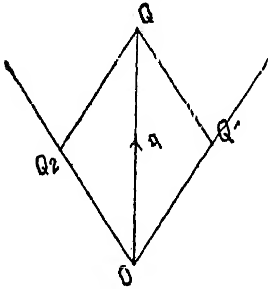


FIG. 8.

For if $\mathbf{q}$ is set off from any point O, terminating at Q, and if through O and Q parallels are drawn to $\mathbf{a}_1$ and $\mathbf{a}_2$ (fig. 8) then these parallels form a parallelogram OQ_1QQ_2. The vectors from O to Q_1 and from Q_1 to Q, since they are parallel to $\mathbf{a}_1$ and $\mathbf{a}_2$, may be represented in the form $x_1\mathbf{a}_1$ and $x_2\mathbf{a}_2$, where the numbers x_1 and x_2 represent the ratio of their lengths to $\mathbf{a}_1$ and $\mathbf{a}_2$ (taking the opposite direction when the ratio is negative). Since together they form the vector $\mathbf{q}$ it follows that:

(3) $$\mathbf{q} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2.$$

We then say that $\mathbf{q}$ is numerically derived from $\mathbf{a}_1$ and $\mathbf{a}_2$.

Accordingly it is to be concluded that all vectors that can be derived numerically from $\mathbf{a}_1$ and $\mathbf{a}_2$ by selecting suitable positive or negative numbers (where one of the two numbers may be zero) are parallel to one and the same plane; and conversely any vector parallel to this plane may be numerically derived from $\mathbf{a}_1$ and $\mathbf{a}_2$.

If three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are of such a nature that no one of them may be derived numerically from the others, then any arbitrary vector $\mathbf{p}$ must be capable of representation in the form

$$(4) \qquad \mathbf{p} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3,$$

where x_1, x_2, and x_3 are positive or negative numbers. To prove this imagine $\mathbf{p}$ drawn from any point O, and terminating at P. Through O pass a plane parallel to the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. If the vector $\mathbf{p}$ lies in this plane then it must be immediately derivable from $\mathbf{a}_1$ and $\mathbf{a}_2$, and x_3 will be zero. If it does not lie in this plane, then P lies outside the plane. Through P draw a parallel to $\mathbf{a}_3$, which must cut the plane, otherwise $\mathbf{a}_3$ would be parallel to the plane, and therefore numerically derivable from $\mathbf{a}_1$ and $\mathbf{a}_2$, contrary to our hypothesis. Let P_1 be the point of intersection, then the vector OP_1 must be numerically derivable from $\mathbf{a}_1$ and $\mathbf{a}_2$, and the vector P_1P from $\mathbf{a}_3$; that is to say, they must be capable of being represented in the respective forms:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \text{ and } x_3\mathbf{a}_3.$$

By adding the two vectors OP_1 and P_1P we have:

$$\mathbf{p} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3.$$

We may consequently assert that from any three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ which cannot be derived numerically from each other, an arbitrary vector may be numerically derived. Imagine the three vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ drawn from a point O and terminating at A_1, A_2, A_3 respectively. Let the vector $\mathbf{p}$ be also drawn from O and terminating at P. The lines OA_1, OA_2, OA_3 we will call the axes of co-ordinates and the planes OA_2A_3, OA_3A_1, OA_1A_2 the co-ordinate planes. If through P three planes are drawn parallel to the co-ordinate planes, they will cut the axes of co-ordinates in the three points P_1, P_2, P_3, and with the co-ordinate planes enclose a parallelepiped, one of whose diagonals is OP. The vectors $x_1\mathbf{a}_1$, $x_2\mathbf{a}_2$, and $x_3\mathbf{a}_3$ lead from O to P_1, P_2, and P_3, and thus consist of three edges of the parallelepiped which meet at a point. The numbers x_α express the ratio of the lengths of these three edges to OA_1, OA_2, and OA_3 and in-

dicate at the same time by their sign whether OP_a is similarly or oppositely directed to OA_a (fig. 9). We term OA_1, OA_2, and OA_3 the unit lengths, the parallelepiped determined by them the unit volume, and the parallelograms fixed by OA_2, OA_3; OA_3, OA_1; and OA_1, OA_2; the unit areas. The numbers x_1, x_2, and x_3 are also called the co-ordinates of the point P in the co-ordinate system OA_1, OA_2, OA_3. They will be referred to here as the coefficients of the vector $\mathbf{p}$ with reference to the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.

Between any four arbitrary vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ there must exist at least one relation of the form:

(5) $$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = 0.$$

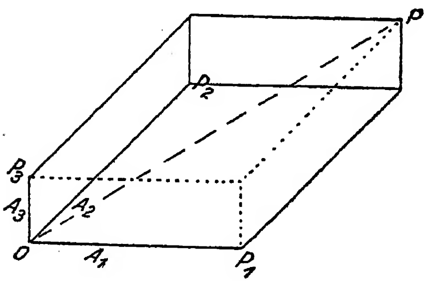


FIG. 9.

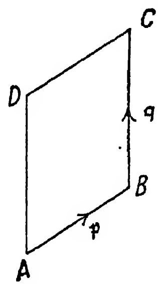


FIG. 10.

For if no such relation holds among any three of them, for example $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ so that in fact x_4 is not zero, then it follows from what has been done, that the vector $-x_4\mathbf{a}_4$, where x_4 may be an arbitrary positive or negative number, must be derivable numerically from $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Hence

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = -x_4\mathbf{a}_4$$

or $$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = 0.$$

§ 7. THE EXTERNAL PRODUCT OF TWO VECTORS

Let $\mathbf{p}$ and $\mathbf{q}$ be two arbitrary vectors. Suppose $\mathbf{p}$ drawn from a point A and terminating at B; and from B suppose $\mathbf{q}$ drawn to C. Now translate the vector $\mathbf{p}$ parallel to itself so that its terminal point runs along BC to C. It will then describe a parallelogram ABCD (fig. 10) on whose boundary

a definite direction ABCDA in the description of the area is given by the direction of the two vectors **p** and **q**. This plane area determined by the two vectors and definitely associated with a particular direction of description we term the *external product* of the two vectors. If the rôles of two vectors are interchanged, so that the vector **q** is first set off from A to D, and then the vector **p** from D to C, the end-point of the vector **q** then being translated along the length DC from D to C, the area ADCBA will then be determined. In this case while the actual content of the parallelogram is the same as before, the direction of description is in the opposite sense. It is necessary to distinguish between these two cases since, as we shall see presently, the sense of description along the boundary plays a very important part. The actual area, moreover, of the figure is of special significance, but the particular shape of the boundary is of no consequence. Any plane area parallel to the vectors **p** and **q** whose content (area) and direction of description is identical with that of the parallelogram ABCDA we may say, is equal to the *external product* of **p** and **q**, and for this the symbol

$$\mathbf{pq}$$

will be utilised. If, on the other hand, the direction of description is reversed, then we say it is equal to the *external product* of **q** and **p** and we write it

$$\mathbf{qp}.$$

A plane element of definite area and sense of description we will call a *vectorial area*. Two *vectorial areas* may then be said to be equal, and only then, when their planes coincide or are parallel, when their areas are equal and when their sense is the same. Vectorial areas we shall indicate by large heavy type. **pq** and **qp** represent opposite vectorial areas, having indeed the same areal content but opposite senses. For every given vectorial area a parallelogram of equal area and sense may be found so that the vectorial area will equal the external product of the two vectors forming two successive sides of the parallelogram. It follows that either of these two vectors, for example the first, may be arbitrarily

chosen, provided it is parallel to the vectorial area. This will fix one side of the parallelogram. The opposite side must then be at such a distance that the area enclosed equals that given, but the side may be moved to any position in the direction of its own length. The given sense of description specifies which of the two possible positions for the vector is to be selected.

§ 8. The Addition of Vectorial Areas

The reason why the vectorial area which originates from the two vectors **p** and **q** is termed a *product* lies in the fact that it obeys certain laws very closely analogous to the laws for the multiplication of two numbers. Nevertheless the laws are not quite the same. This has already become evident from the fact that **p** and **q** are not interchangeable. **pq** and **qp** are not considered equal, but opposite, and their sum is set equal to zero:

(1) $$\mathbf{pq} = -\mathbf{qp}\,;\ \mathbf{pq} + \mathbf{qp} = 0.$$

On the other hand, other analogies certainly exist. If we construct the external product of the sum of two vectors $\mathbf{q}_1 + \mathbf{q}_2$ with a vector **p** lying in the same plane, then from fig. 11 the accuracy of the equation

(2) $$(\mathbf{q}_1 + \mathbf{q}_2)\,\mathbf{p} = \mathbf{q}_1\,\mathbf{p} + \mathbf{q}_2\mathbf{p}$$

is at once evident.*

For the left-hand side of the equation represents the parallelogram ACC′A′ which is traced out by parallel displacement of the vector **p**, along AC, and with the sense of description ACC′A′. On the right-hand side of the equation the first term represents the parallelogram ABB′A′ which is swept out by the vector **p** when its end moves along AB, while the second term represents the parallelogram BCC′B′ which is swept out by the motion of BC along the vector **p**. In actual fact the area of the parallelogram ACC′A′ is equal to the sum of the areas ABB′A′ and BCC′B′.

* It must be noticed that figs. 11, 12, 13, 14 are not for the moment to be regarded as representing projections of spatial figures, but merely plane figures.

It is merely necessary to translate BB′ along its own length so that B comes into the straight line AC (to $\bar{B}$, fig. 11), in which case the two external products on the right-hand side will not alter; B′ will then fall in C′A′ (becoming the point $\bar{B}'$, fig. 11) and the parallelogram ACC′A′ separates into two portions with the same direction sense along the boundaries. The one portion is equal to $\mathbf{q}_1\mathbf{p}$ and the other to $\mathbf{q}_2\mathbf{p}$.

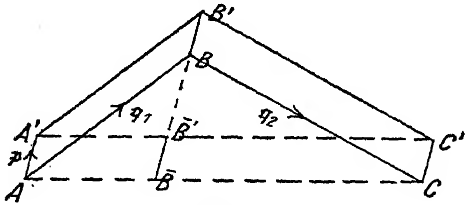


FIG. 11.

The special case may arise where the straight line BB′ cuts the straight line AC when produced (figs. 12 and 13); the two external products $\mathbf{q}_1\mathbf{p}$ and $\mathbf{q}_2\mathbf{p}$ are then of opposite signs. The parallelogram ACC′A′ has in that case the same sense as the larger of the two parallelograms ABB′A′ and BCC′B′. By translating BB′ along its own length, these

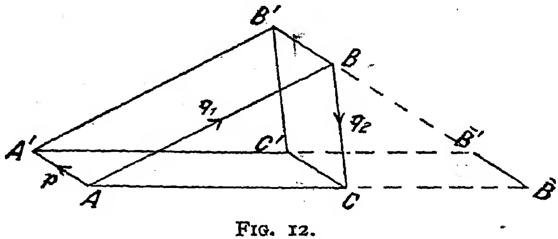


FIG. 12.

transform into A$\bar{B}\bar{B}'$A′ and $\bar{B}$CC′$\bar{B}'$. The larger of these two, for example the parallelogram A$\bar{B}\bar{B}'$A′ in fig. 12, may be considered as of two portions in which the one has the same area as, but is of opposite sense to, that of $\mathbf{q}_2\mathbf{p}$, so that this portion may be removed. The remaining portion is then identical with ACC′A′. In fig. 13 the larger parallelogram is $\bar{B}$CC′$\bar{B}'$. It breaks up into the two portions $\bar{B}$AA′$\bar{B}'$ and

ACC'A', the former of which is neutralised by $\mathbf{q}_1\mathbf{p}$. All these cases are included in the formula

(3) $$(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{p} = \mathbf{q}_1\mathbf{p} + \mathbf{q}_2\mathbf{p}$$

if only a convention is agreed upon associating sign and sense of description. Which sense is to be taken as positive is quite arbitrary. The formula in question corresponds to the ordinary law in algebra for the removal of brackets:

(4) $$(a_1 + a_2)b = a_1b + a_2b.$$

If the combination of vectors is considered as analogous to the addition of numbers, the new operation may be compared

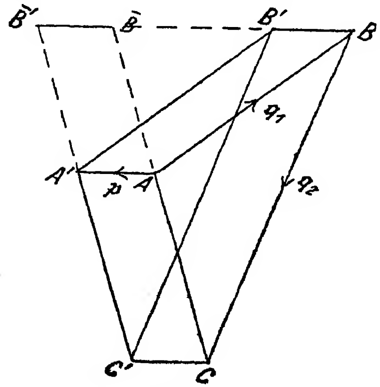


FIG. 13.

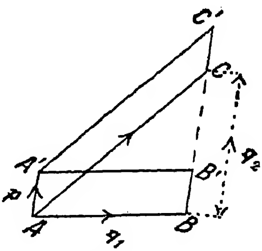


FIG. 14.

with multiplication, even though the analogy is not quite complete. Two similarly or oppositely directed vectors give zero as external product. In this case the sides of the parallelogram fall into one straight line, so that the area vanishes. In this case also the formula

(5) $$(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{p} = \mathbf{q}_1\mathbf{p} + \mathbf{q}_2\mathbf{p}$$

remains true. If for example $\mathbf{q}_1 + \mathbf{q}_2$ is parallel to $\mathbf{p}$, so that ACC'A' vanishes, then BB' will be parallel to the line AA'CC', and the two parallelograms ABB'A' and BCC'B' are of equal area but of opposite sense; $\mathbf{q}_1\mathbf{p}$ and $\mathbf{q}_2\mathbf{p}$ thus neutralise each other.

If q_2 is similarly or oppositely directed to p then q_2p vanishes and therefore

$$(6) \qquad (q_1 + q_2)p = q_1p,$$

or in words, the parallelogram ACC′A′ may be transformed into ABB′A′ without alteration of its area or sense by sliding the portion CC′ along its own direction.

The formula

$$(7) \qquad (q_1 + q_2)p = q_1p + q_2p$$

may also be written in the form

$$(8) \qquad p(q_1 + q_2) = pq_1 + pq_2.$$

For if the "factors" are interchanged in each external product on both sides, the result is the same as if both sides had been multiplied by -1, thus not affecting the truth of the equation.

If any two vectorial areas are to be added together, i.e. if a vectorial area is to be found whose sense of description is the same as that of the larger of the two vectorial areas and whose area equals the sum or difference of the areas of the two given vectorial areas according as their senses are the same or opposite, then this can be effected by representing the two given vectorial areas in the form of two external products

$$q_1p \text{ and } q_2p,$$

in which one factor, for example the second, is common to both. Then

$$(q_1 + q_2)p$$

is equal to the required vectorial area.

Two vectorial areas which are not parallel to the same plane may be combined according to the following law to form a third vectorial area. Since the two vectorial areas are not parallel, any two planes parallel to them will meet. Let p be a vector parallel to the line of intersection, then, as we have seen above, the two areas may be brought into the form:

$$q_1p \text{ and } q_2p.$$

Construct the sum $\mathbf{q}_1 + \mathbf{q}_2$ and call the vectorial area which equals

$$(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{p},$$

the sum of the two given vectorial areas. The formula

$$(\mathbf{q}_1 + \mathbf{q}_2)\mathbf{p} = \mathbf{q}_1\mathbf{p} + \mathbf{q}_2\mathbf{p},$$

which we had to prove for the addition of parallel vectorial areas since in that case addition already had a meaning, may now be utilised to express what we propose in this case to mean by sums and additions of vectorial areas. The justification of the term is again to be found, as in the addition of vectors, in the corresponding laws for addition of numbers which the operations obey.*

If $\mathbf{F}_1$ and $\mathbf{F}_2$ are two vectorial areas, then:

$$\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_2 + \mathbf{F}_1. \tag{9}$$

For, instead of $\mathbf{F}_1$ and $\mathbf{F}_2$ we may put $\mathbf{q}_1\mathbf{p}$ and $\mathbf{q}_2\mathbf{p}$, and then

$$\mathbf{q}_1\mathbf{p} + \mathbf{q}_2\mathbf{p} = (\mathbf{q}_1 + \mathbf{q}_2)\mathbf{p},$$
$$\mathbf{q}_2\mathbf{p} + \mathbf{q}_1\mathbf{p} = (\mathbf{q}_2 + \mathbf{q}_1)\mathbf{p},$$

but since

$$\mathbf{q}_1 + \mathbf{q}_2 = \mathbf{q}_2 + \mathbf{q}_1$$

the terms on the right-hand sides must be equal.

§ 9. The Representation of Vectorial Areas

Let $\mathbf{F}$ be an arbitrary vectorial area and let us associate with it a particular vector $\mathbf{f}$ which bears the following relation to $\mathbf{F}$. In the first place, let $\mathbf{f}$ be perpendicular to $\mathbf{F}$; this admits of two possible directions for $\mathbf{f}$ and in some way we must discriminate between them for the present purpose. Let us then consider a straight line meeting $\mathbf{F}$ perpendicularly at some point. If the boundary of the element of area be described in the given sense, then the straight line will be encircled, and a definite direction of rotation will be associated with the straight line. Let the vector $\mathbf{f}$ hav the direction that a right-hand screw would take if its axis

* The relation between $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{p}$ and the corresponding areas is now to be represented by figures 11, 12, 13, when the latter are regarded as figures in space.

coincided with the straight line and it were rotated in the given direction. Finally, in order to fix the length of **f** uniquely, a definite length is selected as unit in terms of which all lengths are to be expressed and all areas in terms of that area formed by a square whose side is unit length. The length of the vector **f** is then to be determined by the number which expresses the size of the area **F** in terms of the selected unit area. If **F** is given, **f** is thus uniquely determined, and conversely if **f** is given **F** is uniquely determined. For from a knowledge of **f** we can fix a plane at right angles to **f** and a definite sense in this plane corresponding to the rotation of a right screw which advances in the direction of **f**, and finally from the size of **f** in terms of the unit length the area of **F** is fixed in terms of the unit area. It is important to remark that the uniqueness of the relation between **f** and **F** is only obtained by maintaining a constant unit of length and of area. If, for example, the unit of length is doubled and thereby the unit of area quadrupled then the number expressing the size of **F** will be $\frac{1}{4}$ of what it was. Hence the corresponding vector will be only $\frac{1}{2}$ as large as before since the unit of length has been doubled, but the new number specifying this length will be $\frac{1}{4}$ as large as in the first case.

The vector **f** we will term the *representation* * of **F**, and similarly the vectorial area **F** we will term the *representation* * of **f**; this relation will be expressed by means of a vertical line in front of **f** or **F** as the case may be.

Thus:

$$(1) \qquad \mathbf{f} = \mid \mathbf{F} \text{ and } \mathbf{F} = \mid \mathbf{f}.$$

If two vectorial areas $\mathbf{F}_1$ and $\mathbf{F}_2$ are parallel to the same plane, then their representations $\mathbf{f}_1 = \mid \mathbf{F}_1$ and $\mathbf{f}_2 = \mid \mathbf{F}_2$ are also parallel to the same line; moreover, $\mathbf{f}_1 + \mathbf{f}_2$ is evidently the representation of $\mathbf{F}_1 + \mathbf{F}_2$. The latter statement still remains true when $\mathbf{F}_1$ and $\mathbf{F}_2$ are not parallel to the same plane. For imagine $\mathbf{F}_1$ and $\mathbf{F}_2$ lie in two planes which intersect in a straight line, then an arbitrary vector **p** may be taken in this line, and as indicated above two

* Or *complement*.

vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ perpendicular to $\mathbf{p}$ may be determined such that:

$$\mathbf{F}_1 = \mathbf{p}\mathbf{q}_1 \text{ and } \mathbf{F}_2 = \mathbf{p}\mathbf{q}_2.$$

The length of $\mathbf{p}$ may be taken as that of the predetermined unit of length, so that the magnitudes of $\mathbf{q}_1$ and $\mathbf{q}_2$ are equal to those of the areas $\mathbf{F}_1$ and $\mathbf{F}_2$. Then $\mathbf{f}_1$ and $\mathbf{f}_2$ may be derived from $\mathbf{q}_1$ and $\mathbf{q}_2$ by rotating the whole figure by 90° about the line of intersection. The direction of rotation is then that of a right-handed screw which advances in the direction of $\mathbf{p}$. In fig. 15 the vector $\mathbf{p}$ is to be regarded as perpendicular to the plane of the paper and directed away from the reader. It will then be at once recognised that the vector $\mathbf{f}_1 + \mathbf{f}_2$ is obtained by the rotation from $\mathbf{q}_1 + \mathbf{q}_2$. The vector $\mathbf{f}_1 + \mathbf{f}_2$ is consequently perpendicular to the area $\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{p}(\mathbf{q}_1 + \mathbf{q}_2)$, and the magnitude of $\mathbf{f}_1 + \mathbf{f}_2$ is equal to the magnitude of $\mathbf{q}_1 + \mathbf{q}_2$ and accordingly also equal to that of $\mathbf{F}_1 + \mathbf{F}_2$ and the direction of $\mathbf{f}_1 + \mathbf{f}_2$ is that fixed for the representation $|(\mathbf{F}_1 + \mathbf{F}_2)$. It appears then that $\mathbf{f}_1 + \mathbf{f}_2$ is the representation of $\mathbf{F}_1 + \mathbf{F}_2$, or as we may write it:

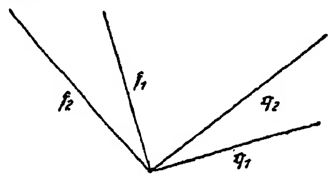


FIG. 15.

(2) $$|(\mathbf{F}_1 + \mathbf{F}_2) = |\mathbf{F}_1 + |\mathbf{F}_2.$$

Conversely

$$\mathbf{F}_1 + \mathbf{F}_2 = |(\mathbf{f}_1 + \mathbf{f}_2),$$

or as we may also write it

(3) $$|(\mathbf{f}_1 + \mathbf{f}_2) = |\mathbf{f}_1 + |\mathbf{f}_2.$$

Otherwise stated, instead of adding two vectorial areas the two vectors representing them may be added and the complement of the vector so obtained may be taken. We deduce from this that the laws of addition of vectors may be applied to the addition of vectorial areas.

If $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$, for example, are three arbitrary vectors, then from § 2, equation (3):

(4) $$(\mathbf{f}_1 + \mathbf{f}_2) + \mathbf{f}_3 = \mathbf{f}_1 + (\mathbf{f}_2 + \mathbf{f}_3).$$

Since the same vector appears on both sides of the equation, the *representation* of the left-hand side equals the *representation* of the right-hand side.

Hence:

$$(5) \qquad |\,[(\mathbf{f}_1 + \mathbf{f}_2) + \mathbf{f}_3] = |\,[\mathbf{f}_1 + (\mathbf{f}_2 + \mathbf{f}_3)].$$

As we have just seen, however, the representation of the sum of two vectors equals the sum of the representations, so that:

$$(6) \qquad |\,(\mathbf{f}_1 + \mathbf{f}_2) + |\,\mathbf{f}_3 = |\,\mathbf{f}_1 + |\,(\mathbf{f}_2 + \mathbf{f}_3).$$

Moreover, according to equation (3)

$$|\,(\mathbf{f}_1 + \mathbf{f}_2) = |\,\mathbf{f}_1 + |\,\mathbf{f}_2,$$

and

$$|\,(\mathbf{f}_2 + \mathbf{f}_3) = |\,\mathbf{f}_2 + |\,\mathbf{f}_3.$$

Writing $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, for the representations $|\,\mathbf{f}_1$, $|\,\mathbf{f}_2$, and $|\,\mathbf{f}_3$, then (6) transforms into:

$$(7) \qquad (\mathbf{F}_1 + \mathbf{F}_2) + \mathbf{F}_3 = \mathbf{F}_1 + (\mathbf{F}_2 + \mathbf{F}_3).$$

The law expressed by equation (7), corresponds exactly to the law expressed in (4) for the combination of vectors, just as the law

$$\mathbf{F}_1 + \mathbf{F}_2 = \mathbf{F}_2 + \mathbf{F}_1$$

corresponds to the law

$$\mathbf{f}_1 + \mathbf{f}_2 = \mathbf{f}_2 + \mathbf{f}_1.$$

From these two laws, as in algebra, the general law follows at once that in the summation of an arbitrary number of terms the result is independent of the order of these terms.

Corresponding to the vector equations considered in § 3, equation (2),

$$\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_r = \mathbf{b}_1 + \mathbf{b}_2 + \ldots + \mathbf{b}_s$$

we obtain equations for the vectorial areas which are the representations

$$(8) \quad |\,\mathbf{a}_1 + |\,\mathbf{a}_2 + \ldots + |\,\mathbf{a}_r = |\,\mathbf{b}_1 + |\,\mathbf{b}_2 + \ldots + |\,\mathbf{b}_s,$$

and conversely from every equation between vectorial areas

we may pass to the corresponding vector equations which hold among their representations.

With this law also it is possible to apply the rule seen to be valid for vectors, that in such an equation a term may be transferred from one side to the other by changing the sign of the term in question. Moreover, in corresponding manner the rules for the multiplication of vectors by positive or negative numbers apply here also for vectorial areas.

§ 10. The Numerical Derivation of a Vectorial Area from Others

Two vectorial areas parallel to the same plane and having the same or opposite senses may be compared by means of a positive or negative number, which gives the ratio of their areas. If the vectorial area **B** is n-times as large as the vectorial area A and if it has the same sense then we will express this fact by means of the equation

$$\mathbf{B} = n\mathbf{A}, \tag{1}$$

while if the senses are opposite

$$\mathbf{B} = -n\mathbf{A}.$$

These equations are equivalent to the vectorial equations

$$|\,\mathbf{B} = n\,|\,\mathbf{A}$$

and

$$|\,\mathbf{B} = -n\,|\,\mathbf{A}$$

which follow from them, and from which they can be deduced. It is merely necessary to remember that the representation of $n\mathbf{a}$ must equal n-times the representation of **a**.

In the same way, from equation (4), § 5,

$$a(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n) = a\mathbf{a}_1 + a\mathbf{a}_2 + \ldots + a\mathbf{a}_n$$

by taking the representations of the terms, it follows that

$$|\,a(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n) = |\,a\mathbf{a}_1 + |\,a\mathbf{a}_2 + \ldots + |\,a\mathbf{a}_n, \tag{2}$$

and hence

$$a\,|\,(\mathbf{a}_1 + \mathbf{a}_2 + \ldots + \mathbf{a}_n) = a\,|\,\mathbf{a}_1 + a\,|\,\mathbf{a}_2 + \ldots + a\,|\,\mathbf{a}_n$$

or

$$a\,|\,(\mathbf{A}_1 + \mathbf{A}_2 + \ldots + \mathbf{A}_n) = a\mathbf{A}_1 + a\mathbf{A}_2 + \ldots + a\mathbf{A}_n. \tag{3}$$

Every vectorial area **P** which is parallel to the same plane as the vectorial area **A** may be derived from it by multiplying it by a positive or negative number:

(4) $$\mathbf{P} = x\mathbf{A}.$$

If $\mathbf{A}_1$ and $\mathbf{A}_2$ are two vectorial areas which are not parallel to the same plane, and if a new vectorial area **P** is derived from these by means of two arbitrary numbers x_1 and x_2 used in the equation

(5) $$\mathbf{P} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2,$$

then there is a straight line parallel to all three vectorial areas. For the intersection of two planes which are parallel to $\mathbf{A}_1$ and $\mathbf{A}_2$ must also be parallel to **P**. The two vectorial areas $x_1\mathbf{A}_1$ and $x_2\mathbf{A}_2$ may be represented by rectangles which have a side in common:

$$x_1\mathbf{A}_1 = \mathbf{a}\mathbf{a}_1,$$
$$x_2\mathbf{A}_2 = \mathbf{a}\mathbf{a}_2.$$

We then have

$$\mathbf{P} = \mathbf{a}\mathbf{a}_1 + \mathbf{a}\mathbf{a}_2 = \mathbf{a}(\mathbf{a}_1 + \mathbf{a}_2),$$

i.e. **P** can be represented by a rectangle of sides **a** and $\mathbf{a}_1 + \mathbf{a}_2$. The three rectangles may be made to correspond to the three faces of a prism with edges parallel to the common vector **a**. The three *representations* of $x\mathbf{A}_1$, $x_2\mathbf{A}_2$, and $-\mathbf{P}$ must then satisfy the equation

$$-\mid\mathbf{P} + \mid x_1\mathbf{A}_1 + \mid x_2\mathbf{A}_2 = 0$$

and therefore when set off in succession one from the extremity of the other, they must form a closed triangle.

Every vectorial area **Q** which is parallel to the common vector of $\mathbf{A}_1$ and $\mathbf{A}_2$ may be presented in the form

(6) $$\mathbf{Q} = x_1\mathbf{A}_1 + x_2\mathbf{A}_2,$$

by a suitable choice of x_1 and x_2. For its *representation* is perpendicular to the common vector, and is consequently parallel to a plane which is parallel to the two *representations* of $\mathbf{A}_1$ and $\mathbf{A}_2$. Hence, as we have seen above, it must be capable of being presented in the form:

$$x_1 \mid \mathbf{A}_1 + x_2 \mid \mathbf{A}_2.$$

The vectorial area itself is consequently:

$$Q = x_1A_1 + x_2A_2.$$

We state this by saying that **Q** is derived numerically from A_1 and A_2. Thus we have the following proposition: All vectorial areas which can be derived numerically from A_1 and A_2 by means of any positive or negative numbers (not excluding the possibility of one of the numbers being zero) are parallel to the same straight line; and conversely every vectorial area parallel to this line may be derived numerically from A_1 and A_2.

If A_1, A_2, and A_3 are three vectorial areas of such a nature that no one of them is derivable numerically from the other two, then any arbitrary vectorial area **P** is capable of being presented in the form

$$P = x_1A_1 + x_2A_2 + x_3A_3, \tag{7}$$

where x_1, x_2, and x_3 are positive, zero, or negative. To prove this it is merely necessary to consider the *representations* of the four vectorial areas. The *representations* of A_1, A_2, and A_3 must be independent of each other in the sense that no one of them can be derived numerically from the other two, for if this were possible for the *representations* it would also be possible for the vectorial areas, contrary to our assumptions. If, however, the *representations* of A_1, A_2, A_3 are independent of each other then, as we have seen above, every vector may be derived numerically from them and consequently also the *representation* of **P**; that is, the numbers x_1, x_2, x_3 may be so selected that:

$$| P = x_1 | A_1 + x_2 | A_2 + x_3 | A_3.$$

But this is equivalent to:

$$P = x_1A_1 + x_2A_2 + x_3A_3.$$

Between any four arbitrary vectorial areas at least one equation of the form

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = 0 \tag{8}$$

must exist. For if between any three of them, say, A_1, A_2, A_3, no such relation holds, i.e. x_4 is not zero, then according

to what has been said, the vectorial area $-x_4A_4$, where x_4 may be any positive or negative number, may be derived numerically from A_1, A_2, A_3, i.e.

$$x_1A_1 + x_2A_2 + x_3A_3 = -x_4A_4,$$

or

$$x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4 = 0.$$

From a point O in space imagine three vectors **a**, **b**, **c** drawn to three arbitrary points A, B, C, the vectors not being co-planar. We will call **A**, **B**, **C** the three vectorial areas formed from the external products **bc**, **ca**, **ab**. They are independent of each other, for the line of intersection of any two is not parallel to the third. Hence all vectorial areas may be presented in the form

$$x_1\mathbf{bc} + x_2\mathbf{ca} + x_3\mathbf{ab},$$

or

$$x_1\mathbf{A} + x_2\mathbf{B} + x_3\mathbf{C},$$

in a manner analogous to the presentation of any vector in the form

$$x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c},$$

where x_1, x_2, x_3 may be positive or negative numbers to be termed the coefficients of the quantity in question.

§ 11. The External Product of a Vectorial Area and a Vector

Let **A** be a vectorial area and **c** an arbitrary vector, then by the external product of **A** with **c** we understand the volume which is described when **A** is displaced by a vectorial amount **c**. We must, however, distinguish carefully the side of **A** towards which the vector **c** is directed. The one side of **A** is reckoned positive, the other negative according to the convention that the vector $|\mathbf{A}$ shall be directed from the negative to the positive direction. Following this we reckon the volume described by **A** in its displacement along the vector **c** as positive or negative, according as **A** is moved towards the positive or negative side. This external product we represent by the expression

$$\mathbf{Ac},$$

or if **A** itself corresponds to the external product of two vectors **a**, **b**, by

abc,

and this we term the external product of

a, **b** and **c**.

Suppose the three vectors **a**, **b**, **c**, drawn from a point O, so that their terminal points are A, B and C; then they constitute three intersecting edges of the parallelepiped whose volume is represented by

abc

with the sign as already explained.

If C lies on the positive side of **ab** then A lies on the positive side of **bc** and B on the positive side of **ca** (fig. 16). And conversely if C lies on the negative side of **ab**, then A and B lie on the negative side of **bc** and **ca** respectively. It appears therefore that we derive exactly the same parallelepiped with the same sign if the vectors are taken in the order **bca** or **cab** as if taken in the order **abc**, that is to say, if the vectorial area **bc** is displaced along **a**, or the vectorial area **ca** along **b**.

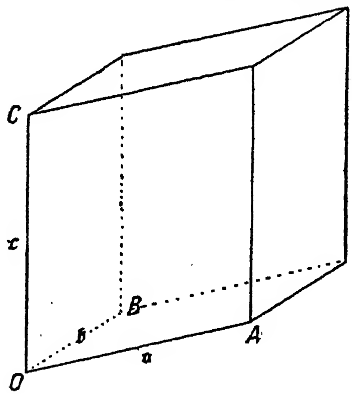


FIG. 16.

Hence:

(1) $$\mathbf{abc} = \mathbf{bca} = \mathbf{cab}.$$

On the other hand

(2) $$\mathbf{bac} = \mathbf{acb} = \mathbf{cba}$$

represents the same parallelepiped but with reversed sign. In other words, the value of

abc

does not alter when **a**, **b**, **c** are interchanged cyclically, but if the positions of two of the vectors are interchanged the sign of the whole is changed.

If the vector **c** is parallel to the vectorial area **A**, then the external product

$$\mathbf{Ac}$$

is zero; for **A** is then displaced in its own plane and therefore describes zero volume.

If in place of **c** the sum of two vectors

$$\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2$$

appears, then

$$(3) \qquad \mathbf{Ac} = \mathbf{A}(\mathbf{c}_1 + \mathbf{c}_2) = \mathbf{Ac}_1 + \mathbf{Ac}_2.$$

As proof consider, in the first place, the special case where $\mathbf{c}_1$ and $\mathbf{c}_2$ are either similarly or oppositely directed, there being no limitation to the generality by also assuming that $\mathbf{c}_1$ and **c** are similarly directed, since $\mathbf{c}_1$ and $\mathbf{c}_2$ may be interchanged. Now imagine the vectorial area **A** translated through $\mathbf{c}_1$ so that the parallelpiped $\mathbf{Ac}_1$ is described, and then displaced again through an amount $\mathbf{c}_2$. This involves adding the parallelepiped $\mathbf{Ac}_2$ where in the first instance $\mathbf{c}_2$ has the same direction as $\mathbf{c}_1$; together they represent **Ac**. In the case where $\mathbf{c}_2$ has the reverse direction, the vectorial area is translated back. In the first translation it describes the parallelepiped $\mathbf{Ac}_1$ in the next $\mathbf{Ac}_2$ with, however, opposite signs, and when added algebraically the parallelepiped **Ac** is derived.

Hence in both cases:

$$\mathbf{Ac} = \mathbf{Ac}_1 + \mathbf{Ac}_2.$$

Now suppose $\mathbf{c}_1$ and $\mathbf{c}_2$ possess any arbitrary directions; initially **A** occupies a certain definite position. It is then displaced through $\mathbf{c}_1$, thus taking up a second position, and then by further displacement through $\mathbf{c}_2$ to a third position which could be obtained by a direct displacement of the vectorial area through the vector **c**. Imagine the second position of the area altered by displacing the latter in its own plane. This affects the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ but leaves $\mathbf{c}_1 + \mathbf{c}_2$

unaltered. The volumes and signs of the parallelepipeds $\mathbf{A}\mathbf{c}_1$ and $\mathbf{A}\mathbf{c}_2$ then remain unaltered also because the two end areas remain in the same planes. The second position may now be so selected that three corresponding points of the first, second and third positions lie in a straight line, i.e. so that $\mathbf{c}_1$ and $\mathbf{c}_2$ are similarly or oppositely directed. Then from what has gone before

$$\mathbf{A}(\mathbf{c}_1 + \mathbf{c}_2) = \mathbf{A}\mathbf{c}_1 + \mathbf{A}\mathbf{c}_2,$$

and since the volume and the sign of the parallelepipeds have not been altered in the change made in the second position, it follows that the same equation holds for arbitrary vectors $\mathbf{c}_1$ and $\mathbf{c}_2$.

Once again insert the external product $\mathbf{ab}$ instead of $\mathbf{A}$ and we get at once:

$$\mathbf{ab}(\mathbf{c}_1 + \mathbf{c}_2) = \mathbf{abc}_1 + \mathbf{abc}_2. \tag{4}$$

Since, as we have already seen, each of the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in the product

$$\mathbf{abc}$$

may be brought to the third position, it follows that we may replace $\mathbf{a}$ and $\mathbf{b}$ in the same way as the sum of two vect in the product.

For example

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$$

$$\mathbf{abc} = \mathbf{bca} = \mathbf{bc}(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{bca}_1 + \mathbf{bca}_2 = \mathbf{a}_1\mathbf{bc} + \mathbf{a}_2$$

Thus

$$(\mathbf{a}_1 + \mathbf{a}_2)\mathbf{bc} = \mathbf{a}_1\mathbf{bc} + \mathbf{a}_2\mathbf{bc} \tag{5}$$

and analogously

$$\mathbf{a}(\mathbf{b}_1 + \mathbf{b}_2)\mathbf{c} = \mathbf{ab}_1\mathbf{c} + \mathbf{ab}_2\mathbf{c}. \tag{6}$$

§ 12. The Connection with the Theory Determinants

Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors whose external product $\mathbf{abc}$ is different from zero and are thus independent of each other, that is to say, no one of them may be numerically derived

from the remainder, then any other vector **p**, as we have already seen, may be numerically derived from the three vectors **a**, **b**, and **c**; so that:

$$\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}.$$

The coefficients x, y, and z may be regarded as the ratios of the external products of three vectors. If the external product **pbc** be constructed and **p** replaced by its expression then we find

$$\mathbf{pbc} = x\mathbf{abc}$$

because **bbc** and **cbc** vanish.

Thus:

$$x = \frac{\mathbf{pbc}}{\mathbf{abc}}.$$

In analogous manner we find:

$$y = \frac{\mathbf{apc}}{\mathbf{abc}},$$

$$z = \frac{\mathbf{abp}}{\mathbf{abc}}.$$

These remarks are closely bound up with the solution of three linear equations with three unknowns. In the calculation of three unknown quantities x, y, z which satisfy the three equations

(1)
$$\begin{aligned} a_1x + b_1y + c_1z &= p_1, \\ a_2x + b_2y + c_2z &= p_2, \\ a_3x + b_3y + c_3z &= p_3, \end{aligned}$$

these three numerical equations are really completely equivalent to one vector equation

(2)
$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{p},$$

where

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3, \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3, \\ \mathbf{c} &= c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3, \\ \mathbf{p} &= p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_3. \end{aligned}$$

For, since $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are numerically independent of each other, a vector equation between them can exist only if all the terms in $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ vanish separately.

The external product **abc** may be multiplied out according to the rules already developed, and this gives:

(4) $$\mathbf{abc} = d\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.$$

The number d we call *the determinant of the three linear equations.* It is constituted from the nine quantities

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}$$

and is composed of the six products corresponding to the six terms that alone remain over in multiplying out:

$$(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)(c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3).$$

Each of these terms accordingly must be a product of three terms, one from each bracket, and if it is not to vanish it must include all three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.

If, for example, $a_2\mathbf{e}_2$ is selected from the first bracket, then either $b_3\mathbf{e}_3$ or $b_1\mathbf{e}_1$ must enter from the second bracket since $b_2\mathbf{e}_2$ multiplied by $a_2\mathbf{e}_2$ would be zero. If, having selected $a_2\mathbf{e}_2$ from the first bracket and $b_3\mathbf{e}_3$ from the second then $c_1\mathbf{e}_1$ must be taken from the third, because $a_2b_3\mathbf{e}_2\mathbf{e}_3$ would be zero if multiplied by either $c_2\mathbf{e}_2$ or $c_3\mathbf{e}_3$. Hence we get the term:

$$a_2b_3c_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1.$$

The six terms that do not vanish in multiplying out the three factors must accordingly all take the form

$$a_\lambda b_\mu c_\nu \mathbf{e}_\lambda \mathbf{e}_\mu \mathbf{e}_\nu,$$

where λ, μ, ν is any one of the six arrangements of the three numbers 1, 2, 3.

Now $$\mathbf{e}_\lambda\mathbf{e}_\mu\mathbf{e}_\nu = \pm\,\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,$$

the positive sign being taken when the arrangement λ, μ, ν is derived from 1, 2, 3 by an even number of interchanges; thus for

123, 231, 312

there would be a positive sign, while for

132, 321, 213

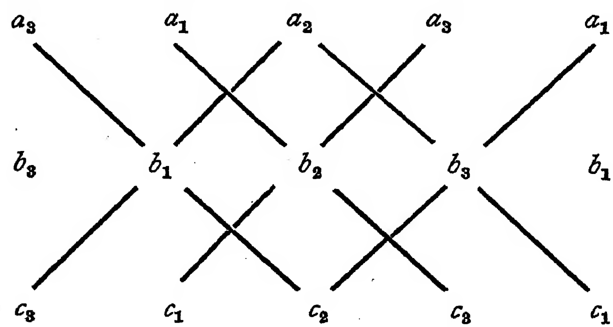
there would be a negative sign. It is in fact equal to the terms

(5) $$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3$$

in the determinant d of the nine quantities:

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{array}$$

In order to calculate the six products quickly with the appropriate sign the following artifice may be utilised. To the three columns of the nine numbers, a column is added with index 1 to the extreme right, and another to the extreme left with index 3. The three products having the positive sign then lie in three lines running inclined from upper left to lower right, while the three products with negative signs, on the other hand, lie on the three inclined lines running from upper right to lower left. Thus:



When the determinant d is different from zero then **a**, **b**, **c** are independent of each other. The unknown quantities x, y, z, as we have seen above, are then uniquely determinate and given by the quotients of the external products:

(6) $$x = \frac{\mathbf{pbc}}{\mathbf{abc}},\ y = \frac{\mathbf{apc}}{\mathbf{abc}},\ z = \frac{\mathbf{abp}}{\mathbf{abc}}.$$

The three external products appearing in the numerators are calculated exactly as with **abc**, except that in one of the horizontal rows the coefficients p_1, p_2, p_3 of the vector **p** appear. Since $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ cuts out from numerator and denominator it follows that x, y, and z are equal to the quotients of the corresponding determinants. If, on the other hand, the determinant d vanishes, then **a**, **b**, **c** are dependent on each other, that is, they are parallel to one plane. If the vector **p** were not parallel to this plane, then it would be impossible to satisfy the vectorial equation

$$\mathbf{a}x + \mathbf{b}y + \mathbf{c}z = \mathbf{p},$$

for the left-hand side can only represent a vector which is parallel to this plane. The three linear equations would then have no solution. To test whether this is the case in any circumstances construct one of the vectorial areas, for example **bc**. If **b** and **c** are not numerically derivable one from the other, then **bc** is not zero. If now, **pbc** differs from zero it follows that **p** is not parallel to the vectorial area **bc** so that the vectorial equation has no solution. If for **b** and **c** their expressions in terms of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are inserted in the vectorial area **bc**, the latter assumes the form

$$\mathbf{bc} = (b_2c_3 - b_3c_2)\mathbf{e}_2\mathbf{e}_3 + (b_3c_1 - b_1c_3)\mathbf{e}_3\mathbf{e}_1 + (b_1c_2 - b_2c_1)\mathbf{e}_1\mathbf{e}_2,$$

so that

$$(7) \qquad \begin{cases} \alpha_1 = b_2c_3 - b_3c_2 \\ \alpha_2 = b_3c_1 - b_1c_3 \\ \alpha_3 = b_1c_2 - b_2c_1 \end{cases}$$

are the three coefficients of the vectorial area, and by means of them it may be derived numerically from the vectorial areas $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2$.

The external product **abc** may then be written

$$(8) \qquad \mathbf{abc} = (a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

so that

$$d = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3,$$

and

$$\mathbf{pbc} = (p_1\alpha_1 + p_2\alpha_2 + p_3\alpha_3)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3.$$

If, on the other hand, it becomes apparent by the vanishing of **pbc** that **p** also is parallel to the vectorial area **bc**, then the vectorial equation

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{p}$$

is soluble for any arbitrary value of x; for the vector $\mathbf{p} - x\mathbf{a}$ is likewise parallel to the vectorial area **bc** and may be derived numerically from **b** and **c** which are assumed independent of each other.

To obtain y for any numerical value of x we select an arbitrary vector **d** which is not derivable numerically from **ab** and **c**, and we construct the external product of both sides of the equation, with **cd**.

Then

$$x\mathbf{acd} + y\mathbf{bcd} = \mathbf{pcd}, \tag{9}$$

from which we have at once

$$y = \frac{\mathbf{pcd}}{\mathbf{bcd}} - x\frac{\mathbf{acd}}{\mathbf{bcd}}, \tag{10}$$

and analogously,

$$z = \frac{\mathbf{pbd}}{\mathbf{cbd}} - x\frac{\mathbf{abd}}{\mathbf{cbd}}.$$

If the vectorial area $\mathbf{bc} = 0$, but **ab** is different from zero, then by an analogous calculation we would find :

$$x = \frac{\mathbf{pbd}}{\mathbf{abd}} - z\frac{\mathbf{cbd}}{\mathbf{abd}},$$

$$y = \frac{\mathbf{pad}}{\mathbf{bad}} - z\frac{\mathbf{cad}}{\mathbf{bad}}.$$

If both the vectorial areas **bc** and **ab** are zero, then all three vectors are parallel to one line. The vector equation

$$\mathbf{a}x + \mathbf{b}y + \mathbf{c}z = \mathbf{p}$$

could not then be satisfied unless **p** also were parallel to that line. This becomes evident by multiplying **p** with any one of the three vectors, for example **a**. Then if the vectorial area **ap** is different from zero, the vector

equation is impossible. If, however, $\mathbf{ap} = 0$, then it can be satisfied by taking perfectly arbitrary values for any two of the three unknowns, say, y and z.

For all four vectors are numerically derivable from any one of them, for example from $\mathbf{a}$. In the same way $\mathbf{p} - y\mathbf{b} - z\mathbf{c}$, where y and z are arbitrary numbers, is numerically derivable from $\mathbf{a}$, its coefficient being x. To determine x, we may utilise any one of the three given linear equations, in which the coefficient of x is not zero, or if we care we may start again from the vector equation :

$$\mathbf{a}x + \mathbf{b}y + \mathbf{c}z = \mathbf{p}.$$

We multiply by any one of the three vectorial areas $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2$. Multiplying by $\mathbf{e}_2\mathbf{e}_3$, for example, we get :

$$x\mathbf{ae}_2\mathbf{e}_3 + y\mathbf{be}_2\mathbf{e}_3 + z\mathbf{ce}_2\mathbf{e}_3 = \mathbf{pe}_2\mathbf{e}_3.$$

But this, however, is nothing more than

$$(xa_1 + yb_1 + zc_1)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = p_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,$$

identical with the first of the system of linear equations.

§ 13. The Scalar Product of Two Vectors

By the scalar product of two vectors $\mathbf{a}$, $\mathbf{b}$ is meant the external product of the one vector with the complement of the other,

$$\mathbf{a} \mid \mathbf{b} \text{ or } \mathbf{b} \mid \mathbf{a},$$

that is to say, the volume (reckoned positive or negative) of the parallelepiped which arises when the vectorial area $\mid \mathbf{b}$ is displaced parallel-wise by the amount represented by the vector $\mathbf{a}$, or what comes to the same thing when the vectorial area $\mid \mathbf{a}$ is displaced in the same manner through $\mathbf{b}$. Imagine $\mathbf{a}$ resolved into two components

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2,$$

where $\mathbf{a}_1$ is perpendicular to $\mid \mathbf{b}$ and $\mathbf{a}_2$ is parallel to $\mid \mathbf{b}$, then from what has gone before

$$\mathbf{a} \mid \mathbf{b} = (\mathbf{a}_1 + \mathbf{a}_2) \mid \mathbf{b} = \mathbf{a}_1 \mid \mathbf{b} + \mathbf{a}_2 \mid \mathbf{b}.$$

Since, however, $\mathbf{a}_2$ is parallel to $|\,\mathbf{b}$, then

$$\mathbf{a}_2 \mid \mathbf{b} = 0,$$

and consequently

(1) $$\mathbf{a} \mid \mathbf{b} = \mathbf{a}_1 \mid \mathbf{b}.$$

Suppose $\mathbf{a}$ and $\mathbf{b}$ lie in the plane of the diagram (fig. 17) so that $|\,\mathbf{b}$ is perpendicular to the plane of the diagram. The height of the parallelepiped is represented by $\mathbf{a}_1$, and the sign is positive or negative according as $\mathbf{a}_1$ has the same or the opposite direction to $\mathbf{b}$, that is, according as the angle ϑ between the vectors $\mathbf{a}$ and $\mathbf{b}$ is acute or obtuse. If $l_{\mathbf{a}}$ and $l_{\mathbf{b}}$ are the numbers determining the lengths of the two vectors, or what is the same thing, for the areas of their representations, then the parallelepiped is represented both in magnitude and in sign by

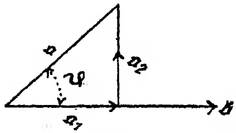


FIG. 17.

$$l_{\mathbf{a}} l_{\mathbf{b}} \cos \vartheta,$$

where at the same time it becomes clear that it is immaterial whether it is the representation of $\mathbf{b}$ that is displaced through $\mathbf{a}$, or the representation of $\mathbf{a}$ through $\mathbf{b}$:

(2) $$\mathbf{a} \mid \mathbf{b} = \mathbf{b} \mid \mathbf{a} = l_{\mathbf{a}} l_{\mathbf{b}} \cos \vartheta.$$

In order to make it clear in the manner of representing the scalar product, that the latter is really directly connected with both vectors, the introduction into it of the complement of either vector being unnecessary and superfluous, we will represent it by:

$$\mathbf{a} \,.\, \mathbf{b}.$$

Thus the scalar product is to be distinguished from the external product by the insertion of a point between the two vectors in the former case.

Thus we deduce from the foregoing:

(3) $$\mathbf{a} \,.\, \mathbf{b} = \mathbf{b} \,.\, \mathbf{a}.$$

If in place of the vector $\mathbf{a}$ we have the sum of two vectors

$$\mathbf{a} = \mathbf{e} + \mathbf{d},$$

then it follows that

$$\mathbf{a} \mid \mathbf{b} = (\mathbf{e} + \mathbf{d}) \mid \mathbf{b} = \mathbf{c} \mid \mathbf{b} + \mathbf{d} \mid \mathbf{b},$$

or in our new presentation

$$(4) \qquad (\mathbf{c} + \mathbf{d}) \, . \, \mathbf{b} = \mathbf{c} \, . \, \mathbf{b} + \mathbf{d} \, . \, \mathbf{b},$$

that is to say, not merely does the communative law $\mathbf{a} \, . \, \mathbf{b} = \mathbf{b} \, . \, \mathbf{a}$ apply to scalar products, but also the distributive law by means of which the brackets may be removed.

If, once more, we replace **d** by the sum of two vectors

$$\mathbf{d} + \mathbf{e},$$

then we have

$$(5) \qquad \begin{aligned} (\mathbf{c} + \mathbf{d} + \mathbf{e}) \, . \, \mathbf{b} &= \mathbf{c} \, . \, \mathbf{b} + (\mathbf{d} + \mathbf{e}) \, . \, \mathbf{b} \\ &= \mathbf{c} \, . \, \mathbf{b} + \mathbf{d} \, . \, \mathbf{b} + \mathbf{e} \, . \, \mathbf{b}, \end{aligned}$$

with analogous results for any number of terms.

The scalar product of two vectors which are mutually perpendicular is zero, for then the one vector is parallel to the representation of the other so that no volume is described during the translation of the vectorial area.

The scalar product of a vector by itself equals the square of the number determining the length of that vector.

§ 14. The Vectorial Product of Two Vectors

By the vectorial product of two vectors, a vector **a** with a vector **b**, is meant that vector **c** which equals the representation of the external product **ab**, thus :

$$\mathbf{c} = \mid \mathbf{ab}.$$

It is therefore directed towards the positive side of the vectorial area **ab**, is at right angles to it, and the magnitude of its length equals the magnitude of the area **ab**, i.e. it is equal to the product of the magnitudes of the lengths of **a** and of **b** into the sine of the contained angle.

Once more, as in the case of the scalar product in order to bring out clearly that the vector **c** arises directly from the vectors **a** and **b** without any superfluous introduction of

the external product, we select as a symbol for the vectorial product :

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}.$$

The presence of the cross between **a** and **b** thus distinguishes the vectorial from the external and from the scalar product. By interchanging **a** and **b** the sense of the external product and with it the representation becomes reversed. Thus we have the rule :

(1) $$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

Moreover, the distributive law holds :

(2) $$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

For, we have already seen that

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$$

and $$|\,(\mathbf{ab} + \mathbf{ac}) = |\,\mathbf{ab} + |\,\mathbf{ac}.$$

Likewise by interchange of the factors :

(3) $$(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{a} \times \mathbf{c}.$$

The external product of two vectors may also be regarded as the complement of the vectorial product **b** × **c** :

$$\mathbf{bc} = |\,(\mathbf{b} \times \mathbf{c}).$$

Hence also the external product of three vectors **a**, **b**, **c** is equal to the scalar product of **a**, **b**, the vectorial product of **b** and **c**:

(4) $$\mathbf{abc} = \mathbf{a}\,|\,(\mathbf{b} \times \mathbf{c}) = \mathbf{a} \,.\, (\mathbf{b} \times \mathbf{c}).$$

The vectorial product of a vector similarly or oppositely directed is zero since their external product is zero.

§—15. The External Product of Two Vectorial Areas

Let A and B be two vectorial areas which we may conceive as lying in two intersecting planes. Let **a** and **b**, moreover, be the complements of **A** and **B**, then the line of

intersection must be perpendicular to both complements. Thus the vector

$$\mathbf{a} \times \mathbf{b}$$

is parallel to the line of intersection and its magnitude is equal to the product of the magnitudes of the areas **A** and **B** into the sine of the angle contained by their two planes.

This vector we call the external product of the vectorial area **A** with the vectorial area **B**. The terminology is justified by the analogy with the external product of two vectors. Just as two vectors, by their external product determine a vectorial area which is parallel to both and whose magnitude equals the product of both lengths into the sine of the angle between them, so two vectorial areas determine a vector which is parallel to both and whose magnitude is equal to the product of the two areas into the sine of the angle between the two vectorial areas. Analogously with the two vectors we represent this external product of **A** and **B** by:

$$\mathbf{AB}.$$

In order to associate in simple manner a particular direction with the vector

$$\mathbf{AB},$$

imagine the vectorial area **A** turned about the line of intersection of the two planes **A** and **B** until its complement has the same direction as that of **B**. The rotation is to be made towards the side where the angle turned through is less than two right angles. The direction of the vector **AB** agrees with the direction of advance of a right-handed screw, whose axis lies along the line of intersection and turns about that line in the same sense as **A**.

The accuracy of this form of presentation follows directly from the definition:

$$\mathbf{AB} = \mathbf{a} \times \mathbf{b}.$$

For in the rotation of **A**, the direction of **a** is changed to that of **b** by the shortest route. The direction of $\mathbf{a} \times \mathbf{b}$ then lies on the positive side of the vectorial area **ab**.

If the planes of **A** and **B** are parallel, then the external product also vanishes because of the vanishing of the sine of the angle between them.

Interchanging **A** and **B**, the sign of the external product is reversed:

$$(1) \qquad \mathbf{BA} = \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b} = -\mathbf{AB}.$$

Moreover, the distributive law is satisfied; for if, for example, **B** is replaced by the sum of two vectorial areas

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2,$$

then from what has already been done, the representation of the sum equals the sum of the representations. Accordingly if we write $\mathbf{b}_1$ and $\mathbf{b}_2$ for the representations of $\mathbf{B}_1$ and $\mathbf{B}_2$, then

$$\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2.$$

and thus

$$(2) \qquad \begin{aligned} \mathbf{AB} &= \mathbf{a} \times (\mathbf{b}_1 + \mathbf{b}_2) = \mathbf{a} \times \mathbf{b}_1 + \mathbf{a} \times \mathbf{b}_2 \\ &= \mathbf{AB}_1 + \mathbf{AB}_2 \end{aligned}$$

Let us imagine the two vectorial areas **A** and **B** to be represented by rectangles having a side in common lying on the line of intersection of the two planes; so that:

$$\mathbf{A} = \mathbf{pr} \qquad \mathbf{B} = \mathbf{qr}.$$

The vector **AB** is then similarly or oppositely directed to **r**, according as **r** has a direction the same as or opposite to the representation of **pq**. For the rotation considered above, by means of which **A** is transformed into the plane of **B**, and the *representation* of **A** turned in the shortest way in the direction of the *representation* of **B**, also turns **p** into the direction of **q**. In other words, **r** has the same direction as **AB** if **pqr** is positive, and the opposite direction if it is negative.

The length of **AB** equals the product of the areas into the sine of the included angle, that is to say, it equals the product of the area **pq** into the square of the length of **r**, or equals the absolute value of **pqr** into the length of **r**. Accordingly also:

$$(3) \qquad (\mathbf{pr})(\mathbf{qr}) = \mathbf{AB} = (\mathbf{pqr})\mathbf{r}.$$

For the right-hand side corresponds to a vector having a direction the same as or opposite to that of $\mathbf{r}$ according to the sign of $\mathbf{pqr}$, and its length has the correct value of the length of $\mathbf{AB}$.

If $\mathbf{A}$ and $\mathbf{B}$ are regarded not as rectangles but as parallelograms with a common side $\mathbf{r}$ so that $\mathbf{p}$ and $\mathbf{q}$ are not at right angles to $\mathbf{r}$, then $\mathbf{p}$ and $\mathbf{q}$ may be written in the form

$$\mathbf{p} = \bar{\mathbf{p}} + \lambda\mathbf{r}$$
$$\mathbf{q} = \bar{\mathbf{q}} + \mu\mathbf{r},$$

where $\bar{\mathbf{p}}$ and $\bar{\mathbf{q}}$ are perpendicular to $\mathbf{r}$ and λ and μ suitable positive or negative numbers. Then

$$\mathbf{A} = \mathbf{pr} = (\bar{\mathbf{p}} + \lambda\mathbf{r})\mathbf{r} = \bar{\mathbf{p}}\mathbf{r}$$
$$\mathbf{B} = \mathbf{qr} = (\bar{\mathbf{q}} + \mu\mathbf{r})\mathbf{r} = \bar{\mathbf{q}}\mathbf{r}\,;$$

and accordingly also

$$(\mathbf{pr})(\mathbf{qr}) = \mathbf{AB} = (\bar{\mathbf{p}}\mathbf{r})(\bar{\mathbf{q}}\mathbf{r}) = (\bar{\mathbf{p}}\bar{\mathbf{q}}\mathbf{r})\mathbf{r}.$$

Now:

$$\bar{\mathbf{p}}\bar{\mathbf{q}}\mathbf{r} = (\mathbf{p} - \lambda\mathbf{r})(\mathbf{q} - \mu\mathbf{r})\mathbf{r} = \mathbf{pqr}.$$

Hence also in this case we have

$$(4) \qquad (\mathbf{pr})(\mathbf{qr}) = (\mathbf{pqr})\mathbf{r},$$

or, remembering that the complements of $\mathbf{A}$ and $\mathbf{B}$ may also be written in the form

$$(5) \qquad (\mathbf{p} \times \mathbf{r}) \times (\mathbf{q} \times \mathbf{r}) = (\mathbf{pqr})\mathbf{r},$$

where $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ may be three arbitrary vectors.

With a suitable choice of $\mathbf{q}$, an arbitrary vector at right angles to $\mathbf{r}$ may be represented by $\mathbf{q} \times \mathbf{r}$. Representing it by $\mathbf{s}$, and remembering that $\mathbf{pqr}$ may also be written in the form $\mathbf{p} \cdot (\mathbf{q} \times \mathbf{r})$, it follows that:

$$(6) \qquad (\mathbf{p} \times \mathbf{r}) \times \mathbf{s} = (\mathbf{p} \cdot \mathbf{s})\mathbf{r}.$$

If $\mathbf{s}$ is not at right angles to $\mathbf{r}$ but is an arbitrary vector, then it may be separated into two parts of which the first is at right angles to $\mathbf{r}$ and the second to $\mathbf{p}$. For this purpose it is merely necessary to consider planes at right angles

to **p** and **r**. If **s** is drawn from a point on the line of intersection of both planes, then through the extremity of **s** draw an arbitrary parallel to one of the planes so that it cuts the other plane. The one component then runs from the origin of **s** in the latter plane to the point of intersection with the parallel, while the other is determined by the parallel itself.

Now writing

$$\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2,$$

where $\mathbf{s}_1$ is perpendicular to $\mathbf{r}$,
and $\mathbf{s}_2$ is perpendicular to $\mathbf{p}$,

then

$$\begin{aligned}(\mathbf{p} \times \mathbf{r}) \times \mathbf{s} &= (\mathbf{p} \times \mathbf{r}) \times (\mathbf{s}_1 + \mathbf{s}_2) \\ &= (\mathbf{p} \times \mathbf{r}) \times \mathbf{s}_1 + (\mathbf{p} \times \mathbf{r}) \times \mathbf{s}_2.\end{aligned}$$

Now from what has gone before

$$(\mathbf{p} \times \mathbf{r}) \times \mathbf{s}_1 = (\mathbf{p} \cdot \mathbf{s}_1)\mathbf{r}$$
$$(\mathbf{p} \times \mathbf{r}) \times \mathbf{s}_2 = -(\mathbf{r} \times \mathbf{p}) \times \mathbf{s}_2 = -(\mathbf{r} \cdot \mathbf{s}_2)\mathbf{p},$$

but

$$\mathbf{p} \cdot \mathbf{s}_1 = \mathbf{p} \cdot (\mathbf{s} - \mathbf{s}_2) = \mathbf{p} \cdot \mathbf{s} - \mathbf{p} \cdot \mathbf{s}_2 = \mathbf{p} \cdot \mathbf{s}$$
$$\mathbf{r} \cdot \mathbf{s}_2 = \mathbf{r} \cdot (\mathbf{s} - \mathbf{s}_1) = \mathbf{r} \cdot \mathbf{s} - \mathbf{r}\mathbf{s}_1 = \mathbf{r} \cdot \mathbf{s}.$$

Accordingly:

$$(\mathbf{p} \times \mathbf{r}) \times \mathbf{s} = (\mathbf{p} \cdot \mathbf{s})\mathbf{r} - (\mathbf{r} \cdot \mathbf{s})\mathbf{p}. \tag{7}$$

This equation still remains true when contrary to our assumption, **p** and **r** are parallel. For then not merely is the left-hand side zero, since $\mathbf{p} \times \mathbf{r}$ is zero, but the right-hand side also vanishes because **p** and **r** are numerically derivable one from the other in the form

$$\mathbf{p} = a\mathbf{r}$$

so that the two sides of (7) vanish simultaneously.

§ 16. Examples, Applications and Exercises

The position of an arbitrary point R is determined by the vector **r** which is drawn from a fixed point O to R. An

arbitrary motion of the point R is given by regarding $\mathbf{r}$ as a function of t. A uniform rectilinear motion, for example, is represented by an equation of the form

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t,$$

where $\mathbf{r}_0$ is the position of the vector at $t = 0$, and $\mathbf{v}$ represents the velocity vector.

Let

$$\mathbf{r}' = \mathbf{r}'_0 + \mathbf{v}'t$$

represent another uniform rectilinear motion. The problem we propose to consider is to determine the shortest distance between the two lines.

For this purpose we construct a vector

$$\mathbf{n} = \mathbf{v} \times \mathbf{v}'$$

which is at right angles to both lines, and reduce it to unit length by dividing it by the square root of its scalar product by itself,

$$\mathbf{e} = \frac{\mathbf{n}}{\sqrt{\mathbf{n} \cdot \mathbf{n}}}.$$

Now choose a vector which joins any one point of the one line to any other point of the other line, e.g.

$$\mathbf{a} = \mathbf{r}' - \mathbf{r}$$

for an arbitrary value of t.

Then $\mathbf{e} \cdot \mathbf{a}$ is the shortest distance.

In fig. 18 imagine both lines parallel to the plane of the diagram, the one drawn heavier lying above the other. The vector $\mathbf{e}$ of unit length is then perpendicular to the plane of the drawing, directed upwards or downwards according to the relative directions of the velocity vectors. If, for example, they lie in the direction of the two arrows then $\mathbf{e}$ is directed upwards and in that case $\mathbf{e} \cdot \mathbf{a}$ is positive and equal to the shortest distance.

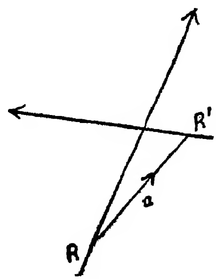


FIG. 18.

If a definite system of co-ordinate axes has been selected, with reference to which the two

motions are presumed to occur, then the vectors must be numerically derivable from the three vectors of the co-ordinate system and the scalar and vector products that arise are to be calculated according to the rules discussed above. To provide a sketch of the method of calculation a numerical example will be worked out for the case of a right-handed co-ordinate system consisting of three mutually perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$:

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t\,;\ \mathbf{r}' = \mathbf{r}_0' + \mathbf{v}'t$$

	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	
$\mathbf{r}_0$:	2	3	− 5	
$\mathbf{r}_0'$:	5	− 6	2	
$\mathbf{r}_0' - \mathbf{r}_0 = \mathbf{a}$:	3	− 9	+ 7	
$\mathbf{v}$:	1	2	6	
$\mathbf{v}'$:	− 1	− 3	1	
$\mathbf{v} \times \mathbf{v}' = \mathbf{n}$:	20	− 7	− 1	$\mathbf{n} \cdot \mathbf{n} = 450.$

$$\frac{\mathbf{n} \cdot \mathbf{a}}{\sqrt{\mathbf{n} \cdot \mathbf{n}}} = \mathbf{e} \cdot \mathbf{a} = \frac{116}{\sqrt{450}}.$$

In the calculation it is advisable, as has been done here, not to write down the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ from which the others are numerically derived, as they occur with each vector, but to arrange to have columns corresponding to them in which the appropriate magnitudes are inserted. In the case of the vector product it is merely necessary to remember that the calculation is conducted according to the distributive law and that here

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

Let $\mathbf{a}$ be a given vector of unit length and a a given positive or negative number, then the equation

$$\mathbf{a}\mathbf{r} = a$$

expresses the condition that the perpendicular projection of

$\mathbf{r}$ on $\mathbf{a}$ has a length a, and has a direction similar or opposite to that of $\mathbf{a}$ according as a is positive or negative.

Consider the vector $\mathbf{r}$ as joining a fixed point O to a variable point R, then

$$\mathbf{a}\,.\,\mathbf{r} = a$$

is the condition that the point R lies in a definite plane at right angles to $\mathbf{a}$, whose distance from O is a units of length, and points from O towards the side indicated by the direction of $\mathbf{a}$ or the opposite, according as a is positive or negative.

If R′ is an arbitrary point not lying on the plane, and the vector joining O to R′, then $\mathbf{r}' - \mathbf{r}$ joins R to R′, and

$$\mathbf{a}\,.\,(\mathbf{r}' - \mathbf{r})$$

is the shortest distance of the point R′ from the plane. By applying the distributive law, we may also write for it:

$$\mathbf{a}\,.\,\mathbf{r}' - a.$$

The expression is positive on that side of the plane towards which the vector $\mathbf{a}$ is directed.

Let a line

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$$

be given external to the plane

$$\mathbf{a}\,.\,\mathbf{r} = a$$

and along which a uniform motion of a point with velocity $\mathbf{v}$ occurs.

At what instant does the point reach the plane?

To determine the value of t we have merely to multiply $\mathbf{r}$ scalarly with $\mathbf{a}$ and to set the product equal to a, then

$$\mathbf{a}\,.\,(\mathbf{r}_0 + \mathbf{v}t) = a$$

or

$$\mathbf{a}\,.\,\mathbf{r}_0 + (\mathbf{a}\,.\,\mathbf{v})\,t = a,$$

so that

$$t = \frac{a - \mathbf{a}\,.\,\mathbf{r}_0}{\mathbf{a}\,.\,\mathbf{v}}.$$

The position of the point of intersection of the straight line with the plane is then given by

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v},$$

where the above value of t is to be inserted.

Let **a**, **b**, **c** be three given vectors numerically independent of each other. Every vector **p** may then be numerically derived from **a**, **b**, **c** in the form

$$\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

and the coefficients x, y, z may be calculated, as we have already seen, by external multiplication with the vectorial areas, **bc**, **ca**, **ab**.

Thus:

$$\begin{aligned} \mathbf{pbc} &= x\mathbf{abc}, \\ \mathbf{pca} &= y\mathbf{abc}, \\ \mathbf{pab} &= z\mathbf{abc}. \end{aligned}$$

Instead of the vectorial areas we will introduce their representations divided by **abc**, and designate these three vectors by $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$:

(1) $$\mathbf{a}^* = \frac{|\,\mathbf{bc}}{\mathbf{abc}},\quad \mathbf{b}^* = \frac{|\,\mathbf{ca}}{\mathbf{abc}},\quad \mathbf{c}^* = \frac{|\,\mathbf{ab}}{\mathbf{abc}},$$

or what is the same thing

$$\mathbf{a}^* = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{abc}},\quad \mathbf{b}^* = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{abc}},\quad \mathbf{c}^* = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{abc}}.$$

Instead of the external product of **p** with $\frac{\mathbf{bc}}{\mathbf{abc}}, \frac{\mathbf{ca}}{\mathbf{abc}}, \frac{\mathbf{ab}}{\mathbf{abc}}$ we may then write the scalar products with $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ so that we have

$$\begin{aligned} \mathbf{p} \cdot \mathbf{a}^* &= x, \\ \mathbf{p} \cdot \mathbf{b}^* &= y, \\ \mathbf{p} \cdot \mathbf{c}^* &= z, \end{aligned}$$

or if we insert these values for x, y, and z,

(2) $$\mathbf{p} = (\mathbf{p} \cdot \mathbf{a}^*)\mathbf{a} + (\mathbf{p} \cdot \mathbf{b}^*)\mathbf{b} + (\mathbf{p} \cdot \mathbf{c}^*)\mathbf{c}.$$

In this manner every vector $\mathbf{p}$ may be expressed in terms of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$. For $\mathbf{p} = \mathbf{a}$ we get:

$$\mathbf{a} = (\mathbf{a} \,.\, \mathbf{a}^*)\mathbf{a} + (\mathbf{a} \,.\, \mathbf{b}^*)\mathbf{b} + (\mathbf{a} \,.\, \mathbf{c}^*)\mathbf{c}.$$

Consequently

$$\begin{aligned} \mathbf{a} \,.\, \mathbf{a}^* &= 1, \\ \mathbf{a} \,.\, \mathbf{b}^* &= 0, \\ \mathbf{a} \,.\, \mathbf{c}^* &= 0, \end{aligned}$$

and similarly

$$\begin{aligned} \mathbf{b} \,.\, \mathbf{b}^* &= 1 \\ \mathbf{c} \,.\, \mathbf{c}^* &= 1 \end{aligned}$$

while all the remaining scalar products of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ vanish.

It is now apparent that the relations between $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ are reciprocal.

If in fact the vectorial products

$$\mathbf{b}^* \times \mathbf{c}^*, \quad \mathbf{c}^* \times \mathbf{a}^*, \quad \mathbf{a}^* \times \mathbf{b}^*$$

are constructed, for example,

$$\mathbf{b}^* \times \mathbf{c}^* = \frac{(\mathbf{c} \times \mathbf{a})}{\mathbf{abc}} \times \mathbf{c}^*,$$

then according to the formula developed above (§ 15, equation 7) we have

$$\begin{aligned} (\mathbf{p} \times \mathbf{r}) \times \mathbf{s} &= (\mathbf{p} \,.\, \mathbf{s})\mathbf{r} - (\mathbf{r} \,.\, \mathbf{s})\mathbf{p} \\ (\mathbf{c} \times \mathbf{a}) \times \mathbf{c}^* &= (\mathbf{c} \,.\, \mathbf{c}^*)\mathbf{a} - (\mathbf{a} \,.\, \mathbf{c}^*)\mathbf{c} = \mathbf{a}, \end{aligned}$$

thus

$$\mathbf{b}^* \times \mathbf{c}^* = \frac{\mathbf{a}}{\mathbf{abc}},$$

and

$$\begin{aligned} \mathbf{a}^* \,.\, \mathbf{b}^* \,.\, \mathbf{c}^* &= \mathbf{a}^* \,.\, (\mathbf{b}^* \times \mathbf{c}^*) \\ &= \frac{\mathbf{a}^* \,.\, \mathbf{a}}{\mathbf{abc}} = \frac{1}{\mathbf{abc}}, \end{aligned}$$

consequently

(3) $$\frac{\mathbf{b}^* \times \mathbf{c}^*}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*} = \mathbf{a},$$

and in analogous manner

$$\frac{\mathbf{c}^* \times \mathbf{a}^*}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*} = \mathbf{b},$$

$$\frac{\mathbf{a}^* \times \mathbf{b}^*}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*} = \mathbf{c}.$$

The same operations consequently which derive $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ from $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, likewise derive $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ from $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$. We thus say that the one system of vectors is reciprocal to the other system. If instead of the system $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we take the system $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{\mathbf{c}}$ where

$$\bar{\mathbf{a}} = n\mathbf{a},\ \bar{\mathbf{b}} = \mathbf{b},\ \bar{\mathbf{c}} = \mathbf{c},$$

then the reciprocal system $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ transforms into

$$\bar{\mathbf{a}}^* = \frac{\bar{\mathbf{b}} \times \bar{\mathbf{c}}}{\mathbf{abc}} = \frac{\mathbf{b} \times \mathbf{c}}{n\mathbf{abc}} = \frac{1}{n}\mathbf{a}^*,$$

$$\bar{\mathbf{b}}^* = \frac{\bar{\mathbf{c}} \times \bar{\mathbf{a}}}{\mathbf{abc}} = \frac{n\mathbf{c} \times \mathbf{a}}{n\mathbf{abc}} = \mathbf{b}^*,$$

$$\bar{\mathbf{c}}^* = \frac{\bar{\mathbf{a}} \times \bar{\mathbf{b}}}{\mathbf{abc}} = \frac{n\mathbf{a} \times \mathbf{b}}{n\mathbf{abc}} = \mathbf{c}^*.$$

Hence if $\mathbf{a}$ is alone altered then in the reciprocal system $\mathbf{a}^*$ is alone altered in inverse proportion. The choice of a unit of length has an effect on the reciprocal relation between the two systems. If for example the one system $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is held fixed and the unit of length is chosen n times as great, then the vector products $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times \mathbf{b}$, as has already been remarked, do not remain unchanged but become $\frac{1}{n}$th of their former lengths. The magnitude of $\mathbf{abc}$ is then $\frac{1}{n^3}$ as large as before. At the same time while $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ have the same directions as before, they are now n^2 as long. By a suitable selection of unit of length we may always arrange that $\mathbf{abc}$ becomes equal to ± 1, by an appropriate selection of the order of the vectors designated by $\mathbf{abc}$ we may arrange so that

$$\mathbf{abc} = 1.$$

We then have also:

$$\mathbf{a}^*\mathbf{b}^*\mathbf{c}^* = 1.$$

When all the vectors

$$p = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

which are numerically derived from **a**, **b**, and **c**, by giving integral values to x, y, and z, are set off from a fixed point O, then the terminal points P form a so-called framework or lattice work in space. The vector joining any point P of the framework to any other point P′ is represented by the difference $\mathbf{p}' - \mathbf{p}$ of the two vectors from O to P′, and from O to P, and may therefore be derived from **a**, **b**, **c** by means of whole numbers. Hence the vector $\mathbf{p}' - \mathbf{p}$ drawn from O must also lead to a lattice point. In other words, the vectors leading from any one lattice point to all the others are the same as those proceeding out from any other lattice point. These we call the vectors of the lattice work.

If **p** is a lattice vector then so also is every vector represented by

$$n\mathbf{p}$$

where n is any positive or negative integer. All points to which one arrives by drawing these vectors from any one lattice point lie on a straight line which is divided into equal intervals by the points. If within this interval other lattice points lie, then there is a lattice vector **q** which has the same direction as **p**, but is not greater than $\mathbf{p}/2$. With this we arrive at another distribution of the straight lines into equal intervals. If again lattice points were to lie within these intervals, then once more they would determine lattice vectors not greater than $\mathbf{q}/2$. This process must come to an end. For if not we would arrive at lattice vectors **s** of arbitrary small length, in which case also their magnitudes $\mathbf{s} \cdot \mathbf{a}^*$, $\mathbf{s} \cdot \mathbf{b}^*$, $\mathbf{s} \cdot \mathbf{c}^*$ in relation to **a**, **b**, **c** must be arbitrarily small without actually all three vanishing. This is not possible, however, since the magnitudes of the vectors are all whole numbers. Thus on each line joining lattice points we have a separation into equal intervals in such a manner that all the lattice points lying on the line are points of separation

between the equal intervals. The lattice vector **u**, corresponding to such an interval, gives, when multiplied by all positive and negative integers, all the lattice vectors parallel to this straight line. Their magnitudes $\mathbf{u} \cdot \mathbf{a}^*$, $\mathbf{u} \cdot \mathbf{b}^*$, $\mathbf{u} \cdot \mathbf{c}^*$ can have no divisor in common, otherwise it would be an integral number of times a smaller lattice vector. Conversely, when their magnitudes have no divisor in common, then all lattice vectors parallel to **u** are integral numbers of times **u**. For if not then **u** would have to be an integral number of times a smaller lattice vector, thus requiring a common divisor.

Imagine now that a straight line parallel to **u** is laid through each lattice point. On each of these lines the successive lattice points occur at intervals **u**. Now through any one lattice point on such a line, and through any lattice point outside the line we pass a plane. The lattice points of this plane may be considered as lying on a system of lines running parallel to **u**, for through each lattice point there runs a straight line parallel to **u**.

Let **p** be a lattice vector of the plane joining one of these lines to any other. The numerical value of the vectorial area **up** is then equal to the product of the length of **u** into the distance apart of the two lines. By constructing the manifold of **p**, we obtain a system of equidistant lattice lines in the plane. If between these there should lie yet other lattice lines, then some one of them is at a distance from one of the lines of the system at most half as great as the distance between two neighbouring lines of the system. It must be possible then to find a lattice vector **q** in the plane such that **uq** is at most half as large as **up**. The whole manifold of **q** then provides a system of equidistant lines parallel to **u**. If still others lie between them then again a lattice vector **s** in the plane may be found for which **us** is at most half as large as **uq**. This process must terminate, for otherwise we would arrive at arbitrary small vectorial areas **us** equal to the external product of two lattice vectors. This is, however, not possible since the coefficients of the vectorial area **us** with reference to **bc**, **ca**, **ab** are equal to

$$\frac{\mathbf{usa}}{\mathbf{abc}}, \quad \frac{\mathbf{usb}}{\mathbf{abc}}, \quad \frac{\mathbf{usc}}{\mathbf{abc}}

and are consequently whole numbers since those of **u** and **s** are such.

One of them must be at least equal to 1 and therefore cannot be part of an arbitrary small **us**. There must then be a lattice vector **v** of such a nature that **uv** is the smallest vectorial area that can be derived by the external product of **u** with a lattice vector in this plane. The lines constituting the whole manifold of these vectors **v** occur at equal intervals and contain all the lattice points in the plane.

All lattice vectors in the plane are then numerically derivable by means of integers from **u** and **v**, thus :

$$n\mathbf{u} + m\mathbf{v}.$$

The lattice points determine a network of parallelograms having **u** and **v** as sides.

We can now show that if through every lattice point planes are drawn parallel to **uv**, then they must be equidistant. Consider then such a plane and an arbitrary lattice vector **p**, joining one of its lattice points to a point outside it. The distance between this plane and one parallel to it through the terminal point of **p** is equal to the numerical value of

$$\mathbf{uvp},$$

divided by the numerical value of **uv**. The whole manifold of vectors **p** leads to a system of such parallel planes with the same distance apart. If now between these planes there lie still other lattice planes, then some one of these planes lies at most half the distance from some plane of the system. A lattice vector **q** leading from the one to the other gives a value (absolute)

$$\mathbf{uvq}$$

which is at most half as large as **uvp**. The whole manifold of vectors **q** leads to a new system of equidistant lattice planes parallel to **uv**. If between these there still lie lattice planes parallel to **uv**, then we must obtain a distance which is at most half as large and a lattice vector **s** for

which $\mathbf{uvs}$ is at most half as large as $\mathbf{uvq}$. This process must terminate, otherwise we would arrive at an indefinitely small value of $\mathbf{uvs}$, but

$$\frac{\mathbf{uvs}}{\mathbf{abc}}$$

is an integer different from zero and consequently at the least its absolute value can be 1. Consequently the absolute value of $\mathbf{uvs}$ is not less than that of $\mathbf{abc}$ and therefore cannot become indefinitely small. There is then a lattice vector $\mathbf{w}$ of such a nature that $\mathbf{uvw}$ has the smallest value which it can have with any lattice vector $\mathbf{w}$ which is not parallel to the vectorial area.

All lattice points are then arranged in planes parallel to $\mathbf{uv}$. The planes are all at equal distances apart such that they are given by the values of $\mathbf{uvw}$ when divided by $\mathbf{uv}$. All lattice vectors are then numerically derivable by means of integral numbers from $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Since $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are also lattice vectors, they are also derivable by means of whole numbers from $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Consequently $\frac{\mathbf{abc}}{\mathbf{uvw}}$ is a whole number just as $\frac{\mathbf{uvw}}{\mathbf{abc}}$. If we write

$$\frac{\mathbf{abc}}{\mathbf{uvw}} = \mathrm{N},$$

then $\frac{1}{\mathrm{N}}$ must also be a whole number, and accordingly

$$\mathrm{N} = \pm 1.$$

The distance apart of the lattice planes parallel to $\mathbf{uv}$ thus equals $\mathbf{abc}$ divided by the numerical value of this vectorial area.

Since every lattice vector is numerically derivable from $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ by means of integral numbers, the vectorial areas $\mathbf{bc}$, $\mathbf{ca}$, $\mathbf{ab}$ are also derivable in the same way from $\mathbf{vw}$, $\mathbf{wu}$, $\mathbf{uv}$. It is necessary merely to insert the values for $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and to multiply out. Incidentally all the vectorial areas of the lattice work, that is to say, all the vectorial areas derivable from $\mathbf{bc}$, $\mathbf{ca}$, $\mathbf{ab}$ by means of whole numbers,

are also derivable in the same way from **vw**, **wu**, **uv**. If such a vectorial area is parallel to the vectorial area **uv** then it must be an integral multiple of **uv**. The coefficients of **uv** can consequently have no divisor in common, and conversely if a vectorial area of the lattice work is parallel to **uv**, and if its coefficients have no divisor in common, then it can differ from **uv** at most only in sign. For otherwise it would be an integral multiple of **uv**, which would necessitate a common divisor in the coefficients.

Every vectorial area **F** of the lattice work

$$\mathbf{F} = \xi\mathbf{bc} + \eta\mathbf{ca} + \zeta\mathbf{ab}$$

where ξ, η, ζ are integers, is parallel to a system of lattice planes. To prove this, imagine a plane parallel to **F** and passing through the lattice point O. Then

$$\frac{\mathbf{F(bc)}}{\mathbf{abc}} = -\eta\mathbf{c} + \zeta\mathbf{b}$$

is a vector parallel to the intersection of this plane with the plane parallel to **bc**, and

$$\frac{\mathbf{F(ca)}}{\mathbf{abc}} = \xi\mathbf{c} - \zeta\mathbf{a}$$

is a vector parallel to the intersection of the plane with a plane parallel to **ca**. Both are lattice vectors which together determine a lattice plane parallel to **F**.

In this lattice plane, as we have already seen, we may determine two lattice vectors **u**, **v**, in terms of which all lattice vectors of the plane may be expressed by means of integral numbers. If then ξ, η, ζ have no factor in common we can arrange the lattice points in equidistant planes parallel to the vectorial area

$$\mathbf{F} = \xi\mathbf{bc} + \eta\mathbf{ca} + \zeta\mathbf{ab},$$

the distance apart being equal to the quotient of the numerical value of **abc** by that of **F**.

The complements of all the vectorial areas **F** of our lattice work likewise constitute the vectors of a lattice work. We will divide them by **abc**.

We may then introduce the vectors reciprocal to **a**, **b**, **c**, viz. **a***, **b***, **c*** and write:

$$\left| \frac{\mathbf{F}}{\mathbf{abc}} = \xi\mathbf{a}^* + \eta\mathbf{b}^* + \zeta\mathbf{c}^*. \right.$$

We term this lattice work *reciprocal* to the original one and vice versa. Every vectorial area of the one lattice work corresponds to a vector perpendicular to it in the other, such that the numerical value of this vector equals the quotient of the numerical value of the vectorial area divided by the external product of its three unit vectors. The value of the reciprocal of this quotient equals the distance between the planes of the one lattice work which stand at right angles to the vectors of the other. Or expressed as a formula:

The distance between the planes of the lattice work **a**, **b**, **c** which are at right angles to the vector

$$\xi\mathbf{a}^* + \eta\mathbf{b}^* + \zeta\mathbf{c}^*,$$

where ξ, η, ζ have no common factor, is equal to

$$\frac{1}{\sqrt{(\xi\mathbf{a}^* + \eta\mathbf{b}^* + \zeta\mathbf{c}^*) \cdot (\xi\mathbf{a}^* + \eta\mathbf{b}^* + \zeta\mathbf{c}^*)}},$$

or the inverse square of the distance equals

$$\xi^2\mathbf{a}^* \cdot \mathbf{a}^* + \eta^2\mathbf{b}^* \cdot \mathbf{b}^* + \zeta^2\mathbf{c}^* \cdot \mathbf{c}^* + 2\eta\zeta\mathbf{b}^* \cdot \mathbf{c}^* + 2\zeta\xi\mathbf{c}^* \cdot \mathbf{a}^* + 2\xi\eta\mathbf{a}^* \cdot \mathbf{b}^*.$$

The coefficients of ξ^2, η^2, ζ^2 are the squares of the lengths of the vectors **a***, **b***, **c***. The coefficients of $2\eta\zeta$, $2\zeta\xi$, $2\xi\eta$ are the products of two lengths into the cosine of the included angle. Accordingly, if we know the six coefficients then we know the relative positions of the three vectors. We can consequently construct the lattice work **a***, **b***, **c*** and hence also its reciprocal, apart from the fact that the whole figure may still be arbitrarily turned or translated in space.

On these considerations a method may be developed for investigating the lattice structure of crystals by means of the interference of Röntgen rays. If light of wave length λ

proceeds from a point A, and falling on a group of material points P converts these to light sources which scatter the light in all directions, then at any point B any considerable degree of light will become apparent only when a sufficient number of waves of equal phase arrive at B simultaneously. Thus the paths AP + PB must have the same length or they must differ from each other by a whole multiple of the wave length λ, that is to say, for a large number of points P, P′,

$$AP' + P'B - AP - PB = \pm n\lambda,$$

where n is a whole number.*

If now the distance PP′ is so small compared with AP and PB that the difference in direction between AP and AP′ and also between PB and P′B may be neglected, then to a sufficient degree of approximation

$$\begin{aligned} AP' - AP &= \mathbf{p} \cdot \mathbf{n}_e, \\ P'B - PB &= -\mathbf{p} \cdot \mathbf{n}_a, \end{aligned}$$

where $\mathbf{p}$ represents the vector PP′, $\mathbf{n}_e$ the vector of unit length giving the direction of the incident ray, and $\mathbf{n}_a$ the corresponding vector for the emergent ray (fig. 19).

The condition for the existence of considerable light at B then becomes :

$$\begin{aligned} &\mathbf{p} \cdot \mathbf{n}_e - \mathbf{p} \cdot \mathbf{n}_a \\ &= \mathbf{p} \cdot (\mathbf{n}_e - \mathbf{n}_a) \\ &= \pm n\lambda. \end{aligned}$$

Representing the angle between the incident and the emergent rays by θ then the length of the vector $\mathbf{n}_e - \mathbf{n}_a$ is equal to $2 \sin \theta/2$.

Accordingly we may write

$$\mathbf{n}_e - \mathbf{n}_a = 2 \sin \theta/2 \cdot \mathbf{s},$$

where $\mathbf{s}$ is a vector of length unity (fig. 20).

* Strictly speaking, it is not necessary that the path differences should be integral multiples of the wave length. Two waves will also strengthen each other when the path difference is less than $\frac{1}{4}\lambda$.

The condition then assumes the form:

$$\mathbf{p} \cdot \mathbf{s} = \pm n \frac{\lambda}{2 \sin \theta/2}.$$

Now on fixing P for a group of points P′, $\mathbf{p} \cdot \mathbf{s} = 0$ implies that all these points P′ lie in the same plane as P, $\mathbf{p} \cdot \mathbf{s} = c$ implies that all points P′ lie in a plane at right angles to $\mathbf{s}$ and at distance c from P. (When c is positive it is on the $\mathbf{s}$ side, for c negative, on the opposite side.)

Thus $$\mathbf{p} \cdot \mathbf{s} = \pm nc$$

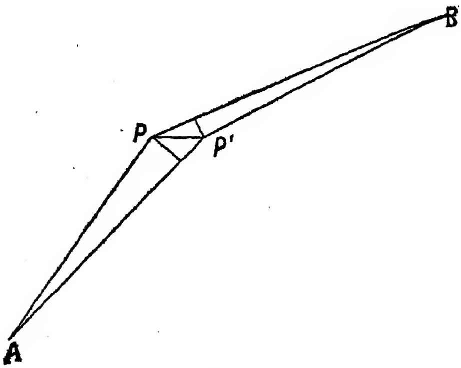


FIG. 19.

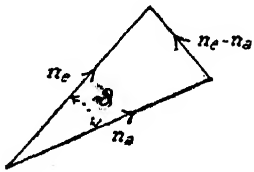


FIG. 20.

implies that the points P′ can be arranged in equidistant planes, distant c apart.

If the points P, P′ constitute a lattice work then, as we have seen, the arrangement into equidistant planes is possible in a variety of ways.

For the lattice work

$$\mathbf{p} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

we have seen that the distances between the equidistant planes were given by the reciprocal of the numerical value of the vector

$$\mathbf{q} = \xi\mathbf{a}^* + \eta\mathbf{b}^* + \zeta\mathbf{c}^*$$

where ξ, η, ζ were three arbitrary positive or negative

integers without any factor in common. The vector **q** is thus perpendicular to the corresponding equidistant planes and is thus parallel to **s**.

The condition

$$\mathbf{p} \cdot \mathbf{s} = \pm \frac{n\lambda}{2 \sin \theta/2}$$

then indicates that

$$\frac{1}{\sqrt{\mathbf{q} \cdot \mathbf{q}}} = \pm \frac{n\lambda}{2 \sin \theta/2},$$

or that

$$n^2 \mathbf{q} \cdot \mathbf{q} \lambda^2 = 4 \sin^2 \theta/2.$$

For $\mathbf{q} \cdot \mathbf{q}$ we may write the quadratic form in ξ, η, ζ and the multiplication by n^2 may be expressed by removing the restriction from ξ, η, ζ that they shall have no factor in common; that is to say; in the equation

$$(\mathbf{a}^* \cdot \mathbf{a}^* \xi^2 + \mathbf{b}^* \cdot \mathbf{b}^* \eta^2 + \mathbf{c}^* \cdot \mathbf{c}^* \zeta^2 + 2\mathbf{b}^* \cdot \mathbf{c}^* \eta\zeta + 2\mathbf{c}^* \cdot \mathbf{a}^* \zeta\xi \\ + 2\mathbf{a}^* \cdot \mathbf{b}^* \xi\eta)\lambda^2 = 4 \sin^2 \theta/2$$

ξ, η, ζ may be any positive or negative numbers.

If the problem deals only with a system of lattice points in one plane, then only the condition

$$\mathbf{p} \cdot \mathbf{s} = 0$$

comes into question. This is the condition for ordinary reflection at a plane.

Then in the equation

$$\mathbf{p} \cdot \mathbf{s} = \pm \frac{n\lambda}{2 \sin \theta/2}$$

we would have $n = 0$ and the angle of reflection would remain arbitrary.

By experiment the angle θ between the incident and the emergent light is measured for all possible orientations of the lattice work (or space grating) with reference to the incident light, for which the light emerges with marked

intensity. From the angle θ and the wave length λ the numerical values of

$$\mathbf{a}^*\xi + \mathbf{b}^*\eta + \mathbf{c}^*\zeta$$

possible for this grating are determined. From this, the constitution $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ of the grating is found and hence also that of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Exercises

Let four points R_1, R_2, R_3, R_4 be specified by the four vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$ radiating out from the point O.

Construct the three vectors

$$\mathbf{a} = \mathbf{r}_2 - \mathbf{r}_1, \quad \mathbf{b} = \mathbf{r}_3 - \mathbf{r}_1, \quad \mathbf{c} = \mathbf{r}_4 - \mathbf{r}_1$$

and show that the distance of the point R_4 from the plane $R_1R_2R_3$ equals

$$\frac{\mathbf{c}' \cdot \mathbf{c}}{\sqrt{\mathbf{c}' \cdot \mathbf{c}'}}$$

where

$$\mathbf{c}' = \mathbf{a} \times \mathbf{b}.$$

Determine the significance of a positive and negative sign for $\mathbf{c}' \cdot \mathbf{c}$ as regards the relative positions of the points R_1, R_2, R_3, R_4.

Show, moreover, that

$$\frac{\mathbf{c}' \cdot \mathbf{c}}{\sqrt{(\mathbf{b}' + \mathbf{c}') \cdot (\mathbf{b}' + \mathbf{c}')}},$$

where $\mathbf{b}' = \mathbf{c} \times \mathbf{a}$ represents the shortest distance between the lines R_1R_2 and R_3R_4.

Imagine the points R_1, R_2 R_3, R_4 given by rectangular co-ordinates with O as origin (all expressed with the same unit of length), then the co-ordinates are at the same time coefficients by means of which the four vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$ are derived from the three unit vectors of the co-ordinate system. For any special choice of co-ordinates carry through the calculation.

For example :

$\mathbf{r}_1$:	1	− 1	2
$\mathbf{r}_2$:	− 1	2	1
$\mathbf{r}_3$:	2	− 1	3
$\mathbf{r}_4$:	3	1	− 5

Hence :

$\mathbf{a}$:	− 2	3	− 1
$\mathbf{b}$:	1	0	1
$\mathbf{c}$:	2	2	− 7

Hence :

$\mathbf{a}'$:	− 2	9	− 2
$\mathbf{b}'$:	19	16	10
$\mathbf{c}'$:	3	1	− 3

Verify :

$$\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 29$$

$\mathbf{b}' + \mathbf{c}'$:	22	17	7

$$\frac{\mathbf{c} \cdot \mathbf{c}'}{\sqrt{\mathbf{c}' \cdot \mathbf{c}'}} = \frac{29}{\sqrt{19}};$$

$$\frac{\mathbf{c}' \cdot \mathbf{c}}{\sqrt{(\mathbf{b}' + \mathbf{c}') \cdot (\mathbf{b}' + \mathbf{c}')}} = \frac{29}{\sqrt{822}}.$$

CHAPTER II

THE DIFFERENTIATION AND INTEGRATION OF VECTORS AND VECTORIAL AREAS

§ 1. Rules for Differentiation

LET the motion of a point R in space be specified by a vector $\mathbf{r}$ regarded as a function of the time t, the vector leading from a fixed point O to the moving point R. We may, in fact, suppose that the vector can be numerically derived from $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, three independent vectors, and that the coefficients are given functions of the time.

We then understand by

$$\frac{d\mathbf{r}}{dt}$$

the vector whose coefficients with respect to the same three vectors are the derivatives with respect to t of the coefficents of $\mathbf{r}$, that is to say, it is the vector of the velocity. It is the limiting value to which the differential coefficient

$$\frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r} - \mathbf{r}_1}{t - t_1}$$

approaches when t and t_1 coalesce. $\mathbf{r} - \mathbf{r}_1$ is the vector leading from the position of the point at time t_1 to the position at time t. Without much further consideration many of the rules of the differential calculus may be applied at once to the differentiation of vectors and their products. The proofs are so simple, if the coefficients of the vectors are dealt with, that it is unnecessary to carry them out here. The rules themselves, however, will be explicitly stated in order to impress them on the memory.

The differential coefficient of a sum is equal to the sum of the differential coefficients of the separate terms

$$\frac{d(\mathbf{p} + \mathbf{q})}{dt} = \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{q}}{dt},$$

and likewise for the sum of any number of vectors.

A constant factor may be taken outside the sign of differentiation:

$$\frac{d(a\mathbf{p})}{dt} = a\frac{d\mathbf{p}}{dt}.$$

If the factor is also a function of t, then the differentiation of the product follows the same rule as with the product of two functions:

$$\frac{d(a\mathbf{p})}{dt} = \frac{da}{dt}\mathbf{p} + a\frac{d\mathbf{p}}{dt}.$$

The same rule applies also to the scalar and to the vector product of two vectors:

$$\frac{d(\mathbf{p}\,.\,\mathbf{q})}{dt} = \frac{d\mathbf{p}}{dt}\,.\,\mathbf{q} + \mathbf{p}\,.\,\frac{d\mathbf{q}}{dt}$$

$$\frac{d(\mathbf{p} \times \mathbf{q})}{dt} = \frac{d\mathbf{p}}{dt} \times \mathbf{q} + \mathbf{p} \times \frac{d\mathbf{q}}{dt}.$$

The proof, instead of being carried through by means of the coefficients, may also be effected directly from a consideration of differences:

$$\begin{aligned}\mathbf{p}\,.\,\mathbf{q} - \mathbf{p}_1\,.\,\mathbf{q}_1 = \mathbf{p}\,.\,\mathbf{q} - \mathbf{p}_1\,.\,\mathbf{q} + \mathbf{p}_1\,.\,\mathbf{q} - \mathbf{p}_1\,.\,\mathbf{q}_1 \\ = (\mathbf{p} - \mathbf{p}_1)\,.\,\mathbf{q} + \mathbf{p}_1\,.\,(\mathbf{q} - \mathbf{q}_1),\end{aligned}$$

or if Δ represents *difference*

$$\Delta(\mathbf{p}\,.\,\mathbf{q}) = \Delta\mathbf{p}\,.\,\mathbf{q} + \mathbf{p}\,.\,\Delta\mathbf{q},$$

and consequently

$$\frac{\Delta(\mathbf{p}\,.\,\mathbf{q})}{\Delta t} = \frac{\Delta\mathbf{p}}{\Delta t}\,.\,\mathbf{q} + \mathbf{p}_1\,.\,\frac{\Delta\mathbf{q}}{\Delta t}.$$

If now $\frac{\Delta\mathbf{p}}{\Delta t}$ and $\frac{\Delta\mathbf{q}}{\Delta t}$ differ sufficiently little from $\frac{d\mathbf{p}}{dt}$ and $\frac{d\mathbf{q}}{dt}$,

and $\mathbf{p}_1$ from $\mathbf{p}$, then the right-hand side differs by as little as we please from:

$$\frac{d\mathbf{p}}{dt} \cdot \mathbf{q} + \mathbf{p} \cdot \frac{d\mathbf{q}}{dt}.$$

In other words, this is the limiting value to which $\frac{\Delta(\mathbf{p} \cdot \mathbf{q})}{\Delta t}$ approaches as t limits to t_1. A similar consideration applies to $\mathbf{p} \times \mathbf{q}$. In actual fact the proof depends essentially on the distributive law, where we have only to write

$$\mathbf{p} \times \mathbf{q} - \mathbf{p}_1 \times \mathbf{q} = (\mathbf{p} - \mathbf{p}_1) \times \mathbf{q},$$

and

$$\mathbf{p}_1 \times \mathbf{q} - \mathbf{p}_1 \times \mathbf{q}_1 = \mathbf{p}_1 \times (\mathbf{q} - \mathbf{q}_1).$$

In the case of the proof by means of the vector coefficients essentially the same laws are applied. For the distributive law shows itself in the fact that the coefficients of the product are linear functions of the coefficients of each single vector. The differential coefficient of a vectorial area is defined in exactly similar manner to that of a vector. Let the coefficients by means of which a vectorial area is derived numerically from three given vectorial areas be regarded as functions of t. Then the vectorial area itself is a function of t, and its differential coefficient is the vectorial area whose coefficients are the derivatives of the three functions which are the coefficients of the original vectorial area.

If the vector $\mathbf{a}$ is the representation of a vectorial area A, then, as we have already seen, $\mathbf{a}$ has the same coefficients as A when it is numerically derived from the three vectors which are the representations of the vectorial areas from which A is derived. From this it follows that, referred to the same three vectors, the differential coefficient of $\mathbf{a}$ has the same coefficients as the differential coefficient of A. In other words, $\frac{dA}{dt}$ and $\frac{d\mathbf{a}}{dt}$ are representations of each other.

For the differentiation of vectorial areas and their products with other vectorial areas or with vectors there are likewise a series of rules into the truth of which it is

scarcely necessary to enter in detail. It is sufficient to write down the propositions

$$\frac{d(\mathbf{A} + \mathbf{B})}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$$

and similarly for any arbitrary number of terms.

If $\quad \mathbf{A} = \mathbf{pq},$

then

$$\frac{d\mathbf{A}}{dt} = \frac{d\mathbf{p}}{dt}\mathbf{q} + \mathbf{p}\frac{d\mathbf{q}}{dt},$$

$$\frac{d(\mathbf{Ap})}{dt} = \frac{d\mathbf{A}}{dt}\mathbf{p} + \mathbf{A}\frac{d\mathbf{p}}{dt},$$

$$\frac{d(\mathbf{pqr})}{dt} = \frac{d\mathbf{p}}{dt}\mathbf{qr} + \mathbf{p}\frac{d\mathbf{q}}{dt}\mathbf{r} + \mathbf{pq}\frac{d\mathbf{r}}{dt},$$

$$\frac{d(\mathbf{AB})}{dt} = \frac{d\mathbf{A}}{dt}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dt}.$$

If in the formula for the differentiation of the scalar product of two vectors

$$\frac{d(\mathbf{p} \cdot \mathbf{q})}{dt} = \frac{d\mathbf{p}}{dt} \cdot \mathbf{q} + \mathbf{p} \cdot \frac{d\mathbf{q}}{dt}$$

the two vectors are set equal to each other, then for the scalar product of a vector with itself we get

$$\frac{d(\mathbf{p} \cdot \mathbf{p})}{dt} = 2\mathbf{p} \cdot \frac{d\mathbf{p}}{dt}.$$

In the differential calculus when we speak of indefinitely small quantities we mean that they are small quantities whose magnitudes are conceived of as decreasing arbitrarily; their relationship to other simultaneously diminishing quantities are then studied. In similar manner we will speak of indefinitely small vectors, the coefficients of which are differentials.

If a vector $\mathbf{r}$ is given as a function of a variable t then

the change produced in $\mathbf{r}$ by increasing t by an amount Δt can, by applying Taylor's series, be represented by:

$$\Delta \mathbf{r} = \frac{d\mathbf{r}}{dt}\Delta t + \frac{d^2\mathbf{r}}{dt^2}\frac{\Delta t^2}{2\,!} + \ldots .$$

This vectorial equation merely combines together the three Taylor's series which give the change produced in the coefficients when expanded in powers of Δt. In the combined form, each term of the Taylor's series represents a vector which enters with the others into a process of vector addition. An analogous form applies to the vectorial area $\mathbf{F}$ regarded as a function of a variable t, in which case also we obtain a Taylor's series in the form:

$$\Delta \mathbf{F} = \frac{d\mathbf{F}}{dt}\Delta t + \frac{d^2\mathbf{F}}{dt^2}\frac{\Delta t^2}{2\,!} + \ldots .$$

§ 2. Curvature and Torsion of a Curve in Space

Suppose a vector $\mathbf{r}$ is drawn from a fixed point O to a point R which varies as a function of t. If t is taken to represent time, then R corresponds to a point moving in space.

The vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

represents the speed of R in magnitude and direction. The absolute value of the speed $\mathbf{v}$ is derived from the equation

$$v^2 = \mathbf{v}\,.\,\mathbf{v}.$$

The vector

$$\mathbf{t} = \frac{\mathbf{v}}{v}$$

has the same direction as $\mathbf{v}$ but is of unit length. It is termed the direction vector of the curve.

We have:

$$\mathbf{t}\,.\,\mathbf{t} = \frac{\mathbf{v}\,.\,\mathbf{v}}{v^2} = 1.$$

It depends merely on the shape of the curve and on the direction in which it is described, that is to say, it is unaltered by changing the speed of description of the same curve.

The vector

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d\mathbf{v}}{dt}$$

is termed the acceleration vector and measures the rate of change of the velocity vector with time.

The vectorial area

$$\mathbf{F} = \mathbf{v}\frac{d\mathbf{v}}{dt}$$

is parallel to the osculating plane of the curve at the point in question. Its complement is the binormal.

If in this expression for the vectorial area $\mathbf{F}$ the direction vector $\mathbf{t}$ is introduced instead of the velocity vector $\mathbf{v}$, then:

$$\mathbf{v} = v\mathbf{t}$$

$$\frac{d\mathbf{v}}{dt} = v\frac{d\mathbf{t}}{dt} + \frac{dv}{dt}\mathbf{t}.$$

Constructing the external product

$$\mathbf{v}\frac{d\mathbf{v}}{dt} = \mathbf{v}\left(v\frac{d\mathbf{t}}{dt} + \frac{dv}{dt}\mathbf{t}\right),$$

then on removing the bracket the second term on the right vanishes since

$$\mathbf{vt} = 0.$$

Hence

$$\mathbf{F} = v^2\mathbf{t}\frac{d\mathbf{t}}{dt},$$

or introducing the arc s instead of the time t,

$$\mathbf{F} = v^3\mathbf{t}\frac{d\mathbf{t}}{ds}.$$

The vector $\frac{d\mathbf{t}}{ds}$ is termed the vector of curvature. It measures the change in the direction vector "per unit of arc." The numerical value of $\frac{d\mathbf{t}}{ds}$ is called the curvature of the curve at the point in question. If the direction vector $\mathbf{t}$ be drawn from a fixed point O terminating at a point T, then during the motion T will move on the surface of a sphere of radius unity and $d\mathbf{t}$ will be the element of arc of the curve described by T on the sphere. The vector $\frac{d\mathbf{t}}{dt}$ is the velocity vector of the moticn of T.

Since

$$\mathbf{t} . \mathbf{t} = 1$$

and consequently on differentiating, since

$$\mathbf{t} . \frac{d\mathbf{t}}{dt} = 0,$$

it follows that the velocity vector of $\mathbf{t}$ is perpendicular to $\mathbf{t}$.

Representing the curvature, that is, the numerical value of $\frac{d\mathbf{t}}{ds}$ by k, then

$$\frac{1}{k}\frac{d\mathbf{t}}{ds}$$

is a vector of unit length, parallel to the osculating plane, perpendicular to the direction vector and pointing to the side to which the path curves. This vector we term the Principal Normal and denote it by $\mathbf{n}$.

Thus:

$$\frac{d\mathbf{t}}{ds} = k\mathbf{n}.$$

The vectorial area may then be written in the form

$$\mathbf{F} = v^3\mathbf{t}\frac{d\mathbf{t}}{ds} = v^3k\mathbf{tn},$$

where $\mathbf{tn}$ may be represented by a square of unit area lying

in the plane of osculation. The numerical value of $\mathbf{F}$ is then equal to

$$v^3 k.$$

The *torsion* of the curve is measured by the rotation of the plane of osculation measured per unit of length of arc.

If

$$\mathbf{F} \text{ and } \mathbf{F} + d\mathbf{F}$$

are the vectorial areas corresponding to two points distant ds apart, then the external product of the two vectorial areas is a vector lying in their line of intersection, the numerical value being equal to the product of their numerical values into the sine of the angle between them.

Thus the external product divided by the product of their numerical values, that is to say

$$\mathbf{F}(\mathbf{F} + d\mathbf{F}) = \mathbf{F}d\mathbf{F}$$

divided by $v^6 k^2$, represents a vector of magnitude equal to the infinitely small angle between the neighbouring osculating planes.

Now:

$$\mathbf{F} = v^3 \mathbf{t} \frac{d\mathbf{t}}{ds}.$$

Hence

$$\frac{d\mathbf{F}}{ds} = 3v^2 \frac{dv}{ds} \mathbf{t} \frac{d\mathbf{t}}{ds} + v^3 \frac{d\mathbf{t}}{ds}\frac{d\mathbf{t}}{ds} + v^3 \mathbf{t} \frac{d^2\mathbf{t}}{ds^2}$$

$$= \frac{3}{v}\frac{dv}{ds} \mathbf{F} + v^3 \mathbf{t} \frac{d^2\mathbf{t}}{ds^2},$$

since

$$\frac{d\mathbf{t}}{ds}\frac{d\mathbf{t}}{ds} = 0.$$

Thus the required vector is:

$$\frac{\mathbf{F}d\mathbf{F}}{v^6 k^2} = \frac{ds}{k^2}\left(\mathbf{t}\frac{d\mathbf{t}}{ds}\right)\left(\mathbf{t}\frac{d^2\mathbf{t}}{ds^2}\right) = \frac{ds}{k^2}\left(\mathbf{t}\frac{d\mathbf{t}}{ds}\frac{d^2\mathbf{t}}{ds^2}\right)\mathbf{t}.$$

The indefinitely small angle between neighbouring osculating planes is then equal to

$$\frac{ds}{k^2}\left(\mathbf{t}\,\frac{d\mathbf{t}}{ds}\,\frac{d^2\mathbf{t}}{ds^2}\right),$$

so that the angle turned through per unit of length is

$$\frac{1}{k^2}\left(\mathbf{t}\,\frac{d\mathbf{t}}{ds}\,\frac{d^2\mathbf{t}}{ds^2}\right).$$

The torsion is here reckoned positive when $\mathbf{F}(\mathbf{F}+d\mathbf{F})$ has the same direction as $\mathbf{t}$, that is to say, when the rotation of the osculating plane takes place about the direction vector $\mathbf{t}$ in the sense of a right-handed screw, for s increasing. Instead of the vectorial area $\mathbf{F}$, its *representation* might equally well be introduced, and the torsion defined by the change in direction of the binormal. Instead of the external product of the two neighbouring vectorial areas we would deal in that case with the vectorial product of the two neighbouring binormals. In essence the two methods are the same.

As an example consider the curvature and torsion of a screw or helix. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three mutually perpendicular vectors of equal length, then a helix is given by the position vector:

$$\mathbf{r} = a\cos\alpha\cos t\,\mathbf{a} + a\cos\alpha\sin t\,\mathbf{b} + a\sin\alpha t\,\mathbf{c}.$$

The vector $\mathbf{r}$ may be considered as composed of the two portions

$$a\cos\alpha\cos t\,\mathbf{a} + a\cos\alpha\sin t\,\mathbf{b}$$

and

$$a\sin\alpha t\,\mathbf{c},$$

of which the first represents a vector drawn from O to the points on a circle, where t is the angle which the vector makes with the initial position $t = 0$. With increasing t it turns from the initial position $a\cos\alpha\,\mathbf{a}$, through 90° into the position $a\cos\alpha\,\mathbf{b}$, and so on in the same sense. The second portion $a\sin\alpha t\,\mathbf{c}$ is at right angles to the plane of the circle and increases proportionately to t, so that the terminal

point of **r** moves on a circular cylinder. If the surface of the cylinder is developed into a plane, the curve becomes a straight line with abscissa $a \cos at$ and ordinate $a \sin at$ if the lengths of the vectors **a**, **b**, and **c** are taken as unity. The angle of slope of the helix is therefore a.

Accordingly we have the following:

	a	**b**	**c**
$\mathbf{r}:$	$a \cos a \cos t$	$a \cos a \sin t$	$a \sin at$
$\mathbf{v} = \dfrac{d\mathbf{r}}{dt}:$	$-a \cos a \sin t$	$a \cos a \cos t$	$a \sin a$

$$\mathbf{v} \cdot \mathbf{v} = v^2 = a^2.$$

Hence:

	a	**b**	**c**	
$\dfrac{\mathbf{v}}{a} = \mathbf{t}:$	$-\cos a \sin t$	$\cos a \cos t$	$\sin a$	$v = \dfrac{ds}{dt} = a$
$\dfrac{d\mathbf{t}}{ds}:$	$-\dfrac{\cos a}{a} \cos t$	$-\dfrac{\cos a}{a} \sin t$	0	$\dfrac{d\mathbf{t}}{ds} \cdot \dfrac{d\mathbf{t}}{ds} = k^2 = \dfrac{\cos^2 a}{a^2}.$
$\dfrac{d^2\mathbf{t}}{ds^2}:$	$\dfrac{\cos a}{a^2} \sin t$	$-\dfrac{\cos a}{a^2} \cos t$	0	

$$\frac{d\mathbf{t}}{ds}\frac{d^2\mathbf{t}}{ds^2} = \frac{\cos^2 a}{a^3}\,\mathbf{ab}$$

$$\mathbf{t}\frac{d\mathbf{t}}{ds}\frac{d^2\mathbf{t}}{ds^2} = \frac{\sin a \cos^2 a}{a^3}\,\mathbf{abc}$$

$$\frac{1}{k^2}\left(\mathbf{t}\frac{d\mathbf{t}}{ds}\frac{d^2\mathbf{t}}{ds^2}\right) = \frac{\sin a}{a}\,\mathbf{abc}.$$

According as the screw is right or left-handed **abc** equals $+1$ or -1.

§ 3. Curvature and Torsion Otherwise Considered

The vectorial area

$$\mathbf{F} = \mathbf{v}\frac{d\mathbf{v}}{dt},$$

which is parallel to the osculating plane, may be defined by the limiting position of three points of the curve when the points ultimately coalesce.

Let
$$\mathbf{r},\ \mathbf{r}_1,\ \mathbf{r}_2$$
be the position vectors of the three points
$$R,\ R_1,\ R_2,$$
and the corresponding values of the variable
$$t,\ t_1,\ t_2,$$
then if $\mathbf{r}_1 - \mathbf{r}$ and $\mathbf{r}_2 - \mathbf{r}$ are represented by $\Delta_1\mathbf{r}$ and $\Delta_2\mathbf{r}$, and if $t_1 - t$ and $t_2 - t$ are represented by $\Delta_1 t$ and $\Delta_2 t$, by Taylor's Theorem

$$\Delta_1\mathbf{r} = \mathbf{v}\Delta_1 t + \frac{d\mathbf{v}}{dt}\frac{\Delta_1 t^2}{2} + \ldots$$

$$\Delta_2\mathbf{r} = \mathbf{v}\Delta_2 t + \frac{d\mathbf{v}}{dt}\frac{\Delta_2 t^2}{2} + \ldots .$$

The vectorial area RR_1R_2 is then equal to:

$$\tfrac{1}{2}\Delta_1\mathbf{r}\Delta_2\mathbf{r} = \frac{\Delta_1 t\Delta_2 t}{2}\left(\mathbf{v} + \frac{d\mathbf{v}}{dt}\frac{\Delta_1 t}{2} + \ \ldots\right)\left(\mathbf{v} + \frac{d\mathbf{v}}{dt}\frac{\Delta_2 t}{2} + \ \ldots\right)$$

$$= \frac{\Delta_1 t\Delta_2 t\,(\Delta_2 t - \Delta_1 t)}{4}\,\mathbf{v}\frac{d\mathbf{v}}{dt} + \ldots .$$

Consequently:

$$\mathbf{F} = \mathbf{v}\frac{d\mathbf{v}}{dt}$$

$$= \underline{\lfloor}\,\frac{2\Delta_1\mathbf{r}\Delta_2\mathbf{r}}{\Delta_1 t\Delta_2 t\,(\Delta_2 t - \Delta_1 t)}.$$

If instead of the quantities $\Delta_1 t$, $\Delta_2 t$ and $(\Delta_2 t - \Delta_1 t)$ the corresponding lengths $\Delta_1 s$, $\Delta_2 s$, $\Delta_3 s$ of the triangle are introduced, then since

$$\underline{\lfloor}\,\frac{\Delta_1 s}{\Delta_1 t} = \underline{\lfloor}\,\frac{\Delta_2 s}{\Delta_2 t} = \underline{\lfloor}\,\frac{\Delta_3 s}{\Delta_2 t - \Delta_1 t} = v,$$

we must have

$$\mathbf{F} = \mathbf{v}\frac{d\mathbf{v}}{dt} = \underline{\lfloor}\,\frac{2\Delta_1\mathbf{r}\Delta_2\mathbf{r}}{\Delta_1 s\Delta_2 s\Delta_3 s}\,v^3.$$

In other words, the vectorial area

$$\frac{\mathbf{F}}{v^3}$$

equals four times the indefinitely small triangle RR_1R_2 divided by the product of the three sides of the triangle. If a, b, c are the three sides of a triangle (fig. 21), and α the angle opposite the side a, then it follows directly from the figure that

$$a = 2\rho \sin \alpha$$

where ρ is the radius of the circumscribing circle.

Thus

$$abc = 2\rho \sin \alpha bc,$$

or since

$$\sin \alpha bc = \text{twice the area of the triangle},$$

$$\frac{1}{s} = \frac{4 \times \text{area of triangle}}{abc}.$$

Accordingly the numerical value of $\mathbf{F}/v^3$ may be defined as the limit value of the reciprocal of the circumscribing radius of the triangle RR_1R_2 when the three points coalesce. In this way also we arrive at the conclusion deduced above that the numerical value of $\mathbf{F}$ is equal to the third power of the velocity multiplied by the curvature.

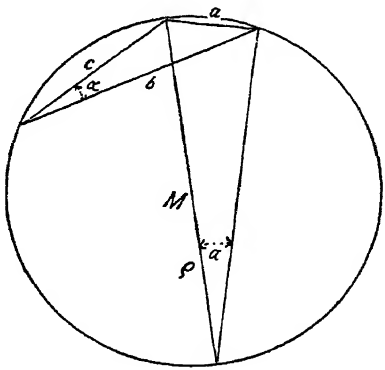


FIG. 21.

Yet another consideration leads to the same result. A distance of length unity is set off along the tangent to the curve at R in the direction of increasing t, terminating at T so that RT is equal to the direction vector $\mathbf{t}$. The vectorial area RTR_1 is then equal to:

$$\tfrac{1}{2}\mathbf{t}\Delta_1\mathbf{r} = \tfrac{1}{2}\mathbf{t}\left(\mathbf{v}\Delta_1 t + \frac{d\mathbf{v}}{dt}\frac{\Delta_1 t^2}{2} + \ldots\right)$$

$$= \mathbf{t}\frac{d\mathbf{v}}{dt}\frac{\Delta_1 t^2}{4} + \ldots .$$

Hence

$$\frac{\mathbf{F}}{v} = \mathbf{t}\frac{d\mathbf{v}}{dt} = \mathop{\mathrm{L}}\frac{2\mathbf{t}\Delta_1\mathbf{r}}{\Delta_1 t^2},$$

or on introducing $\Delta_1 s$ again

$$\frac{\mathbf{F}}{v^3} = \mathop{\mathrm{L}}\frac{2\mathbf{t}\Delta_1\mathbf{r}}{\Delta_1 s^2}.$$

In other words, the vectorial area $\mathbf{F}/v^3$ equal four times the indefinitely small vectorial area divided by the square of the chord RR_1. Since the side RT is of unit length, the area of the triangle equals half the distance of the point R_1 from the tangent RT. Imagine a circle drawn through R and R_1 and tangent to RT, then it is known that RR_1^2 is equal to the product of the diameter into the distance of R_1 from the tangent RT (fig. 22). Four times the area of the triangle RR_1T divided by the square of RR_1 is therefore equal to the reciprocal of the radius ρ; that is to say, the numerical value of $\mathbf{F}/v^3$ equals the limiting value of $1/\rho$ when R_1 and R coalesce, or the numerical value of the vectorial area $\mathbf{F}$ is equal to the cube of the velocity multiplied by the curvature. We can reduce the numerical value to unity by taking the vectorial area:

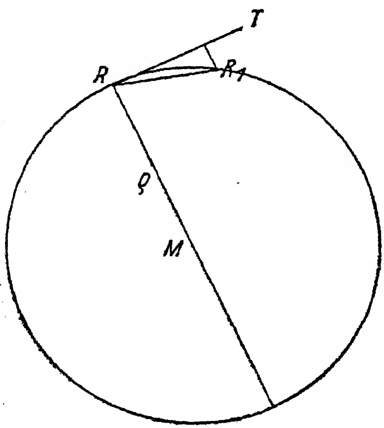


FIG. 22.

$$\frac{\mathbf{F}}{kv^3}.$$

To evaluate the distance of the point R_1 from the osculating plane at R we need merely construct the external product:

$$\frac{\mathbf{F}}{kv^3}\Delta_1\mathbf{r} = \frac{\mathbf{F}}{kv^3}\left(\mathbf{v}\Delta_1 t + \frac{d\mathbf{v}}{dt}\frac{\Delta_1 t^2}{2} + \frac{d^2\mathbf{v}}{dt^2}\frac{\Delta_1 t^3}{6} + \ldots\right)$$
$$= \frac{\mathbf{F}}{kv^3}\frac{d^2\mathbf{v}}{dt^2}\cdot\frac{\Delta_1 t^3}{6} + \ldots .$$

The sign is positive or negative according as R_1 lies on the positive or the negative side of the vectorial area **F** in the osculating plane.

Multiplying the equation by 6 and dividing by $\Delta_1 t^3$, the first term on the right-hand side becomes equal to

$$\frac{1}{kv^3}\left(\mathbf{v}\frac{d\mathbf{v}}{dt}\frac{d^2\mathbf{v}}{dt^2}\right),$$

that is to say, it is the limiting value attained by six times the distance of the point R_1 from the osculating plane, divided by $\Delta_1 t^3$ when R_1 coalesces with R, or if instead of dividing by $\Delta_1 t^3$ we divide by the cube of RR_1 and remember that

$$\underline{\left|\frac{RR_1}{\Delta_1 t}\right.} = v,$$

then

$$\underline{\left|6\frac{SR_1}{(RR_1)^3}\right.} = \frac{1}{kv^6}\left(\mathbf{v}\frac{d\mathbf{v}}{dt}\frac{d^2\mathbf{v}}{dt^2}\right)$$

where S is the foot of the perpendicular from R_1 on the osculating plane (fig. 23), and SR_1 is to be taken as positive or negative according as R_1 lies on the positive or negative side of the vectorial area $\mathbf{F} = \mathbf{v}\dfrac{d\mathbf{v}}{dt}$ considered as lying in the osculating plane. The same sign must be given to RR_1 and $\Delta_1 t$ since $\underline{\left|\frac{RR_1}{\Delta_1 t}\right.}$ is to be the numerical value of **v**. By definition the limit depends only on the shape of the curve. Moreover, it is independent of the direction in which

the curve is described. For if it is reversed then the direction of rotation of the vectorial area $\mathbf{F} = \mathbf{v}\,\frac{d\mathbf{v}}{dt}$ is also changed and thus the positive and negative sides of the osculating plane are interchanged so that the sign of SR_1 is reversed. But at the same time that of RR_1 is also reversed. Hence the expression

$$\frac{1}{kv^6}\left(\mathbf{v}\,\frac{d\mathbf{v}}{dt}\,\frac{d^2\mathbf{v}}{dt^2}\right)$$

is quite independent of the particular choice of the variable t. This may also be seen directly, for let τ be a new

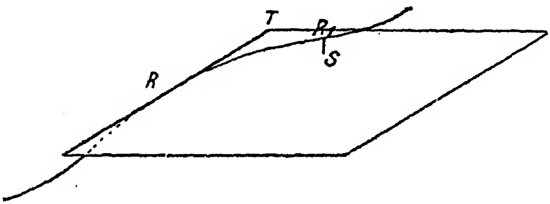


FIG. 23.

variable and t a function of τ, then the velocity vector $\bar{\mathbf{v}}$ referred to the variable τ becomes

$$\bar{\mathbf{v}} = \frac{d\mathbf{r}}{d\tau} = \mathbf{v}\,\frac{dt}{d\tau}$$

and

$$\bar{\mathbf{v}} \cdot \bar{\mathbf{v}} = (\mathbf{v} \cdot \mathbf{v})\left(\frac{dt}{d\tau}\right)^2.$$

Hence

$$\begin{aligned}\bar{\mathbf{F}} = \bar{\mathbf{v}}\,\frac{d\bar{\mathbf{v}}}{d\tau} &= \mathbf{v}\,\frac{dt}{d\tau}\left[\frac{d\mathbf{v}}{dt}\left(\frac{dt}{d\tau}\right)^2 + \mathbf{v}\,\frac{d^2t}{d\tau^2}\right]\\ &= \mathbf{v}\,\frac{d\mathbf{v}}{dt}\left(\frac{dt}{d\tau}\right)^3\\ &= \mathbf{F}\left(\frac{dt}{d\tau}\right)^3;\end{aligned}$$

further:

$$\frac{d^2\bar{\mathbf{v}}}{d\tau^2} = \frac{d^2\mathbf{v}}{dt^2}\left(\frac{dt}{d\tau}\right)^3 + 3\frac{d\mathbf{v}}{dt}\left(\frac{dt}{d\tau}\right)\frac{d^2t}{d\tau^2} + \mathbf{v}\frac{d^3t}{d\tau^3},$$

and

$$\bar{\mathbf{v}}\frac{d\bar{\mathbf{v}}}{d\tau}\frac{d^2\bar{\mathbf{v}}}{d\tau^2} = \mathrm{F}\left(\frac{dt}{d\tau}\right)^3\frac{d^2\bar{\mathbf{v}}}{d\tau^2}$$

$$= \mathrm{F}\frac{d^2\mathbf{v}}{dt^2}\left(\frac{dt}{d\tau}\right)^6,$$

and finally

$$\frac{\bar{\mathbf{v}}\dfrac{d\bar{\mathbf{v}}}{d\tau}\dfrac{d^2\bar{\mathbf{v}}}{d\tau^2}}{(\bar{\mathbf{v}}\,.\,\bar{\mathbf{v}})^3} = \frac{\mathbf{v}\dfrac{d\mathbf{v}}{dt}\dfrac{d^2\mathbf{v}}{dt^2}}{(\mathbf{v}\,.\,\mathbf{v})^3}.$$

If t is chosen to represent the length of the arc, we get:

$$\mathrm{L}\frac{6\mathrm{SR}_1}{(\mathrm{RR}_1)^3} = \frac{1}{k}\left(t\frac{dt}{ds}\frac{d^2t}{ds^2}\right).$$

This is k times the expression already found for the rotation of the osculating plane measured per unit of arc length.

The expression found above, which has the dimensions of an inverse length we will denote by $1/\sigma$,

$$\rho^2\left(t\frac{dt}{ds}\frac{d^2t}{ds^2}\right) = 1/\sigma,$$

and analogously with the expression *radius of curvature* ($k = 1/\rho$ we term σ the *radius of torsion.* In this connection it is to be remarked that ρ has only positive values whereas σ is positive or negative according as the twist of the curve is that of a right-handed or left-handed screw. We propose to investigate the relation between the radius of torsion and the radius of the sphere which has contact of the fourth order at the point of the curve in question. For this purpose let a sphere pass through the circle of curvature of the curve at the point R and through the point R_1.

The position vector of the centre of curvature is

$$\mathbf{r} + \rho\mathbf{n}$$

where

$$\mathbf{n} = \rho\frac{d\mathbf{t}}{ds}$$

represents the principal normal $[(\mathbf{n}\,.\,\mathbf{n}) = 1]$. The binormal we will represent by $\mathbf{b}$, its length, as with that of the principal normal, being 1, thus:

$$\mathbf{b} = \rho \mid \mathbf{t}\frac{d\mathbf{t}}{ds}.$$

The position vector leading to the centre M of the sphere may then be written

$$\mathbf{r} + \rho\mathbf{n} + l\mathbf{b},$$

where l represents the distance, positive or negative, of the centre from the plane of osculation. l is determined from the fact that the vector MR_1 has a length $\sqrt{\rho^2 + l^2}$.

As before, we write

$$\Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}$$

and derive the equation of condition

$$(\Delta\mathbf{r} - \rho\mathbf{n} - l\mathbf{b})\,.\,(\Delta\mathbf{r} - \rho\mathbf{n} - l\mathbf{b}) = \rho^2 + l^2.$$

If the left-hand side be expanded, we must note that:

$$\mathbf{n}\,.\,\mathbf{n} = \mathbf{b}\,.\,\mathbf{b} = 1, \text{ and } \mathbf{n}\,.\,\mathbf{b} = 0.$$

Hence the equation reduces to:

$$\Delta\mathbf{r}\,.\,\Delta\mathbf{r} - 2\Delta\mathbf{r}\,.\,(\rho\mathbf{n} + l\mathbf{b}) = 0.$$

Insert the Taylor's expansion for $\Delta\mathbf{r}$, viz.:

$$\Delta\mathbf{r} = \mathbf{t}\Delta s + \frac{\mathbf{n}}{\rho}\frac{\Delta s^2}{2} + \frac{d^2\mathbf{t}}{ds^2}\frac{\Delta s^3}{6} + \ldots$$

Since $\mathbf{t}\,.\,\mathbf{n} = 0$ we find

$$\Delta\mathbf{r}\,.\,\Delta\mathbf{r} = \Delta s^2 + \text{terms of the fourth order},$$

$$2\rho\Delta\mathbf{r}\,.\,\mathbf{n} = \Delta s^2 + \frac{1}{3}\rho^2\frac{d\mathbf{t}}{ds}\,.\,\frac{d^2\mathbf{t}}{ds^2}\Delta s^3 + \ldots,$$

$$2\Delta\mathbf{r}\,.\,\mathbf{b} = \frac{1}{3}\frac{d^2\mathbf{t}}{ds^2}\,.\,\mathbf{b}\Delta s^3 + \ldots,$$

$$\frac{d^2\mathbf{t}}{ds^2}\,.\,\mathbf{b} = \frac{d^2\mathbf{t}}{ds^2}\,\Big|\,\mathbf{b} = \left(\frac{d^2\mathbf{t}}{ds^2}\,\mathbf{t}\,\frac{d\mathbf{t}}{ds}\right)\rho = \frac{1}{\rho\sigma},$$

$$\frac{d\mathbf{t}}{ds}\,.\,\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{2}\frac{d}{ds}\left(\frac{d\mathbf{t}}{ds}\,.\,\frac{d\mathbf{t}}{ds}\right) = -\frac{d\rho/ds}{\rho^3}.$$

Consequently:

$$+\frac{1}{3}\left(\frac{1}{\rho}\frac{d\rho}{ds}-\frac{1}{\rho\sigma}\right)\Delta s^3+\ .\ .\ .\ =0.$$

For $\Delta s=0$ we then get

$$1=\sigma\frac{d\rho}{ds}$$

so that the square of the radius of the sphere is given by

$$R^2=\rho^2+\sigma^2\left(\frac{d\rho}{ds}\right)^2.$$

§ 4. Rules for Integration

From the differentiation of a vector regarded as a function of one variable, its integration follows. By the integral of a given vector we understand a vector whose differential coefficient equals that of the given vector. The integral is determinate to an arbitrary additive constant vector, which disappears on differentiating:

$$\int \mathbf{r}(t)dt=\mathbf{s}(t)+\mathbf{a}.$$

Integrating from a particular value t_0 of t:

$$\int_{t_0}^{t}\mathbf{r}(t)dt=\mathbf{s}(t)-\mathbf{s}(t_0).$$

The coefficients of the integral of $\mathbf{r}(t)$ with reference to any fixed vectors from which $\mathbf{r}(t)$ can be numerically derived, are consequently equal to the integrals of the coefficients of $\mathbf{r}(t)$. The three arbitrary additive constants may then be grouped together into a constant additive vector.

For the integration of a vector identical rules apply as with the integration of a function; these we need merely state as their proof follows directly from the fact that the integration of the vector is simply a combination of the three integrations of the coefficients.

A constant numerical factor may be taken from under the sign of integration :

$$\int a\mathbf{r}dt = a\int \mathbf{r}dt.$$

The formula for integration by parts may be applied in two ways to the product of a scalar function $f(t)$ with a vector $\mathbf{r}(t)$:

$$\int f(t)\mathbf{r}(t)dt = f(t)\mathbf{s}(t) - \int f'(t)\mathbf{s}(t)dt,$$

where $\mathbf{s}(t)$ is an integral of $\mathbf{r}(t)$, and

$$\int f(t)\mathbf{r}(t)dt = \mathrm{F}(t)\mathbf{r}(t) - \int \mathrm{F}(t)\mathbf{r}'(t)dt,$$

where $\mathrm{F}(t)$ is an integral of $f(t)$.

If

$$\frac{d\mathbf{s}}{dt} = \mathbf{r},$$

so that

$$d\mathbf{s} = \mathbf{r}dt$$

and

$$\int f(t)\mathbf{r}(t)dt = \int f(t)d\mathbf{s},$$

the integral

$$\int f(t)d\mathbf{s}$$

extends over the curve traced out by the end point of $\mathbf{r}$ as t runs through its values, where $\mathbf{r}$ is supposed drawn from a fixed point O.

If $\mathbf{a}$ is a vector independent of t, then

$$\int (\mathbf{a} \,.\, \mathbf{r})dt = \mathbf{a} \,.\int \mathbf{r}dt,$$

and

$$\int (\mathbf{a} \times \mathbf{r})dt = \mathbf{a} \times \int \mathbf{r}dt.$$

If both $\mathbf{a}$ and $\mathbf{r}$ are independent of t then the formula for integration by parts may be applied :

$$\int \left(\mathbf{a} \,.\, \frac{d\mathbf{s}}{dt}\right) dt = \int \mathbf{a} \,.\, d\mathbf{s} = \mathbf{a} \,.\, \mathbf{s} - \int \left(\frac{d\mathbf{a}}{dt} \,.\, \mathbf{s}\right) dt,$$

$$\int \left(\mathbf{a} \times \frac{d\mathbf{s}}{dt}\right) dt = \int \mathbf{a} \times d\mathbf{s} = \mathbf{a} \times \mathbf{s} - \int \left(\frac{d\mathbf{a}}{dt} \times \mathbf{s}\right) dt.$$

For the proof it is merely necessary to remember the corresponding formulæ for the coefficients and to note that the value of $\mathbf{a} \cdot d\mathbf{s}$ and the coefficients of $\mathbf{a} \cdot d\mathbf{s}$ are linear functions of the coefficients of $\mathbf{a}$ and of $d\mathbf{s}$.

§ 5. Application to the Motion of a Point Mass about a Fixed Centre

Let a point R of mass m be in motion under the influence of an attractive or repulsive force from a fixed centre O, the magnitude of the force being a function of the distance from O.

Let the position of the point be specified by a position vector $\mathbf{r}$ set off from O, whose dependence on the time t it is required to determine.

Let the numerical value of $\mathbf{r}$ be represented by r, $(\mathbf{r} \cdot \mathbf{r} = r^2)$ and the vector $\dfrac{\mathbf{r}}{r}$ by $\mathbf{e}$. Let the magnitude of the force be $f(r)$ where a positive value of $f(r)$ signifies a repulsion and a negative value an attraction.

The force vector is then equal to

$$f(r)\mathbf{e}$$

and the differential equation of the motion written as a vector equation runs

$$m\ddot{\mathbf{r}} = f(r)\mathbf{e},$$

where $\ddot{\mathbf{r}}$ represents the second derivative of $\mathbf{r}$ with respect to the time.

Since $$\mathbf{r}\mathbf{e} = 0,$$

external multiplication by $\mathbf{r}$ gives

$$\mathbf{r}\ddot{\mathbf{r}} = 0.$$

The vectorial area $\mathbf{r}\ddot{\mathbf{r}}$, however, is the differential coefficient of the vectorial area $\mathbf{r}\dot{\mathbf{r}}$. Hence by integration $\mathbf{r}\dot{\mathbf{r}}$ equals some constant vectorial area $\mathbf{C}$, or

$$\mathbf{r}d\mathbf{r} = \mathbf{C}dt.$$

$\frac{\mathbf{r}d\mathbf{r}}{2}$ is the vectorial area described by the vector OR during the motion of the point mass in time dt, taken in the sense determined by the order $\mathbf{r}$, $\dot{\mathbf{r}}$.

Hence in an interval t_1 to t_2 a vectorial area

$$\frac{1}{2}\int_{t_1}^{t_2}\mathbf{C}dt = \frac{1}{2}\mathbf{C}(t_2 - t_1)$$

is described. In other words, the motion occurs in a certain plane parallel to C, and equal areas are described by $\mathbf{r}$ in equal times. The area described per unit of time equals $\frac{1}{2}c$, where c represents the numerical value of the vectorial area C.

The vectorial area $\mathbf{r}d\mathbf{r}$ may also be written in the form $r^2\mathbf{e}d\mathbf{e}$. For

$$\mathbf{r} = r\mathbf{e},$$

consequently

$$d\mathbf{r} = rd\mathbf{e} + dr\mathbf{e},$$

so that

$$\mathbf{r}d\mathbf{r} = r\mathbf{e}\,(rd\mathbf{e} + dr\mathbf{e}) = r^2\mathbf{e}d\mathbf{e}.$$

Thus

$$\mathbf{e}d\mathbf{e} = \frac{dt}{r^2}\mathbf{C},$$

or writing $\mathbf{c}$ for the complement of $\mathbf{C}$

$$\mathbf{e} \times \dot{\mathbf{e}} = \frac{1}{r^2}\mathbf{c}.$$

Since

$$\mathbf{e} \,.\, \mathbf{e} = 1$$

and therefore

$$\mathbf{e} \,.\, \dot{\mathbf{e}} = 0$$

it follows that $\dot{\mathbf{e}}$ is at right angles to $\mathbf{e}$. Thus

$$\mathbf{c} \times \mathbf{e} = r^2(\mathbf{e} \times \dot{\mathbf{e}}) \times \mathbf{e} = r^2\dot{\mathbf{e}}.$$

If the initial equation be multiplied vectorially on both sides by $\mathbf{c}$, it now assumes the form

$$m\ddot{\mathbf{r}} \times \mathbf{c} = -\,r^2 f(r)\dot{\mathbf{e}}.$$

This form is equivalent to the first for it transforms back into the original on multiplying vectorially by $-\mathbf{c}/c^2$. In the case where the attraction or repulsion is inversely proportional to the square of the distance

$$f(r) = \frac{\kappa}{r^2},$$

this second form of the differential equation is preferable to the first. For in this case we get

$$m\ddot{\mathbf{r}} \times \mathbf{c} = -\kappa\dot{\mathbf{e}}$$

or

$$\frac{m}{\kappa}\mathbf{c} \times \ddot{\mathbf{r}} = \dot{\mathbf{e}}.$$

Both sides are now differential coefficients, so that on integrating we obtain the vectorial equation

$$\frac{m}{\kappa}\mathbf{c} \times \dot{\mathbf{r}} = \mathbf{a} + \mathbf{e},$$

where $\mathbf{a}$ represents a constant vector, and then by scalar multiplication by $\mathbf{r}$

$$-\frac{m}{\kappa}c^2 = \mathbf{a}\,.\,\mathbf{r} + r,$$

$$[\text{since } (\mathbf{c} \times \dot{\mathbf{r}})\,.\,\mathbf{r} = (\dot{\mathbf{r}} \times \mathbf{r})\,.\,\mathbf{c} = -\,\mathbf{c}\,.\,\mathbf{c} = -\,c^2].$$

These two equations embody the complete representation of the motion. The last equation contains the path in polar co-ordinates while the previous equation gives the shape of the hodograph.

If then the length of the vector $\mathbf{a}$ is represented by ϵ, and the angle between $\mathbf{a}$ and $\mathbf{r}$ by ϕ, the equation to the path may be put in the form

$$-\frac{m}{\kappa}c^2 = \epsilon r \cos\phi + r,$$

or

$$r = \frac{-\,mc^2/\kappa}{1 + \epsilon\cos\phi}.$$

For $\epsilon < 1$ this represents an ellipse and for $\epsilon > 1$ an hyperbola. The focus is situated at the fixed centre O, and the direction $\phi = 0$, i.e. the direction of the vector $\mathbf{a}$ defines the shortest distance. Since r is necessarily positive and in the case of the ellipse $1 + \epsilon \cos \phi$ is also positive, ϵ being less than 1, it follows that $-mc^2/\kappa$ is positive and consequently κ is negative. The elliptic path can only arise with an attractive force. The sum of the maximum and of the minimum distance gives the value of the major axis $2a$:

$$2a = -mc^2/\kappa\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right),$$

$$\therefore\ a = -\frac{mc^2/\kappa}{1-\epsilon^2},$$

whence the equation to the path may be written in the alternative form

$$\frac{r}{a} = \frac{1-\epsilon^2}{1+\epsilon\cos\phi}.$$

We propose to draw a picture of the path diminished in the ratio 1 : a. The major axis then equals 2 and the distance OO′ between the two foci is 2ϵ. The vector O′M leading from the other focus O′ to the centre M equals $\mathbf{a}$. The vector OS is equal to $\mathbf{r}/a$ (fig. 24).

The equation

$$\frac{m}{\kappa}\mathbf{c} \times \dot{\mathbf{r}} = \mathbf{a} + \mathbf{e}$$

which provides the hodograph may be represented by an extension of fig. 24. For this purpose imagine O′ is the fixed centre and let a circle of unit radius be described about M as centre. Let the radius MH be drawn parallel to the vector OS $= \mathbf{r}/a$, so that it represents the vector $\mathbf{e}$. The vector O′H then gives $\mathbf{a} + \mathbf{e}$, that is $\frac{m}{\kappa}\mathbf{c} \times \dot{\mathbf{r}}$. Imagine the plane of the path in fig. 25 regarded from the side indicated by $\mathbf{c}$, so that $\mathbf{r}$ and $\dot{\mathbf{r}}$ rotate in a counter clockwise direction

The vector $\dfrac{\mathbf{c} \times \dot{\mathbf{r}}}{c}$ will then be equal to the vector $\dot{\mathbf{r}}$ rotated a further 90°. Or if the vector O′H is turned through 90° in a clockwise direction we find

$$\frac{m}{\kappa}\frac{\mathbf{c}}{c} \times (\mathbf{c} \times \dot{\mathbf{r}}) = -\frac{m}{\kappa}(\mathbf{c} \times \dot{\mathbf{r}}) \times \frac{\mathbf{c}}{c} = -\frac{mc}{\kappa}\dot{\mathbf{r}}.$$

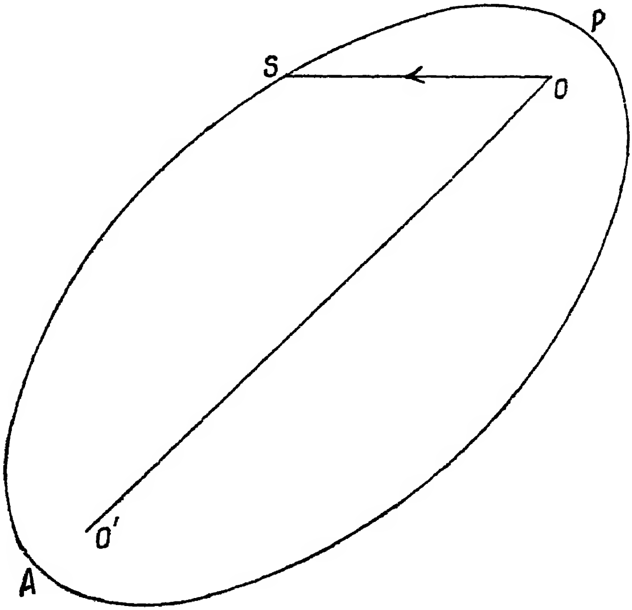


FIG. 24.

Since $-mc/\kappa$ is positive, the vector O′H turned through 90° will indicate the direction of $\dot{\mathbf{r}}$, while its length will be proportional to $\dot{\mathbf{r}}$. We may imagine the scale for representing the velocity so chosen that the unit of length in fig. 25 represents the velocity $-\kappa/mc$. Then O′H in magnitude and direction indicates the velocity to be associated with each point of the path. That O′H is at right angles to $\dot{\mathbf{r}}$ might also have been deduced from the well-known property of an

ellipse that the tangent at S cuts the circle on the major axis at the feet of the perpendicular from the foci on to the tangent. That SH is at right angles to O'H is nothing other than is contained in the above equation:

$$\frac{r}{a} = \frac{1 - \epsilon^2}{1 + \epsilon \cos \phi},$$

for the vector HS is equal to the difference between O'S and O'H

$$2\mathbf{a} + \frac{\mathbf{r}}{a} - (\mathbf{e} + \mathbf{a}),$$

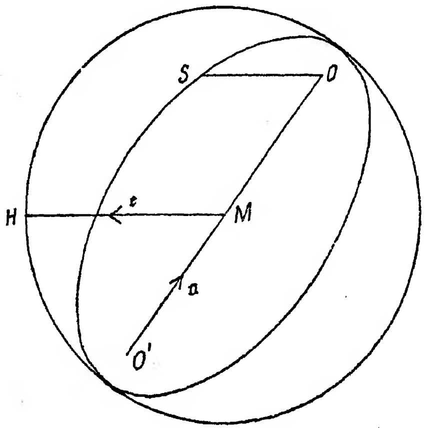


FIG. 25.

or

$$\mathbf{a} + \frac{\mathbf{r}}{a} - \mathbf{e},$$

whose scalar product with $\mathbf{a} + \mathbf{e}$ gives

$$\mathbf{a} \cdot \mathbf{a} - \mathbf{e} \cdot \mathbf{e} + \frac{r}{a}(\mathbf{e} \cdot \mathbf{a}) + \frac{r}{a}(\mathbf{e} \cdot \mathbf{e}),$$

or

$$\epsilon^2 - 1 + \frac{r}{a}\epsilon \cos \phi + \frac{r}{a},$$

which disappears in virtue of the foregoing equation.

§ 6. Surface and Volume Integrals

Just as we have conceived a space curve divided up into vectorial elements and defined a vector by means of an integral

$$\int f\,d\mathbf{r}$$

where $d\mathbf{r}$ was an indefinitely small element of the curve in space and f a scalar function, so we may imagine the surface of a body or a portion of that surface split up into an infinite number of small vectorial areas, and a new vectorial area defined by the integral

$$\int f\,d\mathbf{F},$$

where $d\mathbf{F}$ represents an infinitely small vectorial area and f a position function. We will suppose the *sense* of the vectorial area $d\mathbf{F}$ so chosen that the interior of the body lies on the positive side.

To refer this vectorial area to three given ones it will be necessary to evaluate three double integrals. For instance if $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are three mutually perpendicular vectors of unit length, and x, y, z the coefficients of a position vector $\mathbf{r}$ with reference to $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, then f may be written as a function of x, y, z capable of being represented in terms of any two of these quantities for all points on the surface. The vectorial area $d\mathbf{F}$ may be numerically derived from $|\,\mathbf{i}$, $|\,\mathbf{j}$, $|\,\mathbf{k}$:

$$d\mathbf{F} = dp\,|\,\mathbf{i} + dq\,|\,\mathbf{j} + dr\,|\,\mathbf{k},$$

where

$$d\mathbf{F}\mathbf{i} = dp,$$
$$d\mathbf{F}\mathbf{j} = dq,$$
$$d\mathbf{F}\mathbf{k} = dr$$

are the magnitudes of the projections of $d\mathbf{F}$ on the three coordinate planes. The double integrals

$$\int f dp,\ \int f dq,\ \int f dr,$$

then furnish the three coefficients of

$$\int f d\mathbf{F}.$$

ellipse that the tangent at S cuts the circle on the major axis at the feet of the perpendicular from the foci on to the tangent. That SH is at right angles to O'H is nothing other than is contained in the above equation:

$$\frac{r}{a} = \frac{1 - \epsilon^2}{1 + \epsilon \cos \phi},$$

for the vector HS is equal to the difference between O'S and O'H

$$2\mathbf{a} + \frac{\mathbf{r}}{a} - (\mathbf{e} + \mathbf{a}),$$

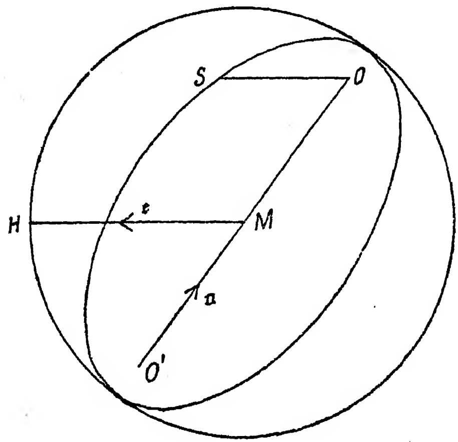


FIG. 25.

or

$$\mathbf{a} + \frac{\mathbf{r}}{a} - \mathbf{e},$$

whose scalar product with $\mathbf{a} + \mathbf{e}$ gives

$$\mathbf{a} \cdot \mathbf{a} - \mathbf{e} \cdot \mathbf{e} + \frac{r}{a}(\mathbf{e} \cdot \mathbf{a}) + \frac{r}{a}(\mathbf{e} \cdot \mathbf{e}),$$

or

$$\epsilon^2 - 1 + \frac{r}{a} \epsilon \cos \phi + \frac{r}{a},$$

which disappears in virtue of the foregoing equation.

§ 7. VECTOR FIELDS AND FIELDS OF VECTORIAL AREAS

A region of space for points of which a vector is defined, is termed a vector field; in the same sense we propose to refer to a field of vectorial area. If $\mathbf{r}$ is a position vector joining a fixed point O to a point R of the region in question, then a scalar function f defined as a function of position for the region is represented in the form $f(\mathbf{r})$, and likewise any vector $\mathbf{p}$ which is a function of position and defines a vector field is represented by $\mathbf{p}(\mathbf{r})$; a vectorial area $\mathbf{P}$ defining a field of vectorial area is represented by $\mathbf{P}(\mathbf{r})$.

A function of position $f(\mathbf{r})$ on differentiation leads to a definite vector field. All points of the space in the neighbourhood of a point R, for which $f(\mathbf{r})$ has the same value as at R constitute a surface to which we assume a tangent plane at R to exist.

Suppose the vector $\mathbf{r}$ derived by means of the coefficients x, y, z from three mutually independent vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}.$$

Then f becomes a function of x, y, z, for which

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

and this may be regarded as the scalar product of the two vectors

$$\mathbf{g} = \frac{\partial f}{\partial x} \mathbf{a}^* + \frac{\partial f}{\partial y} \mathbf{b}^* + \frac{\partial f}{\partial z} \mathbf{c}^*$$

and

$$d\mathbf{r} = dx\mathbf{a} + dy\mathbf{b} + dz\mathbf{c},$$

whereby $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ is meant the reciprocal system to $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{a}^* = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{abc}}, \quad \mathbf{b}^* = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{abc}}, \quad \mathbf{c}^* = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{abc}}.$$

Hence:

$$df = \mathbf{g} \cdot d\mathbf{r}.$$

This vector $\mathbf{g}$ we call the gradient of the scalar function f and it defines a vector field. It is at right angles to the surface $f =$ constant, which passes through R. For df vanishes when $d\mathbf{r}$ and $\mathbf{g}$ are at right angles. If $d\mathbf{r}$ acquires the same direction as $\mathbf{g}$ and is of length dn, then

$$d\mathbf{r} = \frac{dn\mathbf{g}}{\sqrt{\mathbf{g} \cdot \mathbf{g}}}$$

so that

$$df = dn\sqrt{\mathbf{g} \cdot \mathbf{g}},$$

that is to say, the numerical value of the gradient is measured by $\dfrac{df}{dn}$.

By means of the equation

$$df = \mathbf{g} \cdot d\mathbf{r},$$

which is satisfied by the vector $\mathbf{g}$, its independence from the selected unit vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in terms of which it was originally defined, is made evident. For from this equation both its direction and magnitude may be directly derived without reference to the selected unit vectors. In other words, if instead of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we chose any other mutually independent vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ from which the vector $\mathbf{r}$ may be numerically derived,

$$\mathbf{r} = \xi\mathbf{p} + \eta\mathbf{q} + \zeta\mathbf{r},$$

so that f is now to be regarded as a function of ξ, η, ζ, and

$$df = \frac{\partial f}{\partial \xi}d\xi + \frac{\partial f}{\partial \eta}d\eta + \frac{\partial f}{\partial \zeta}d\zeta,$$

then the same vector $\mathbf{g}$ is capable of being expressed in the form

$$\mathbf{g} = \frac{\partial f}{\partial \xi}\mathbf{p}^* + \frac{\partial f}{\partial \eta}\mathbf{q}^* + \frac{\partial f}{\partial \zeta}\mathbf{r}^*$$

where once again $\mathbf{p}^*$, $\mathbf{q}^*$, $\mathbf{r}^*$ stands for the reciprocal system to $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$.

In this derivation of the vector $\mathbf{g}$ from the function of position f use is also made of the symbol

$$\mathbf{g} = \nabla f$$

where ∇ expresses the vectorial operator,

$$\nabla = \frac{\partial}{\partial x}\mathbf{a}^* + \frac{\partial}{\partial y}\mathbf{b}^* + \frac{\partial}{\partial z}\mathbf{c}^*$$

which is applied to the function f. This symbol is derived from the Hebrew word "Nabla."* By the introduction of new independent vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ the operator takes the form

$$\nabla = \frac{\partial}{\partial \xi}\mathbf{p}^* + \frac{\partial}{\partial \eta}\mathbf{q}^* + \frac{\partial}{\partial \zeta}\mathbf{r}^*.$$

Accordingly we may speak of ∇ as a vector whose coefficients referred to $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ are the differential operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ if x, y, z represent the coefficients referred to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. On transforming to new co-ordinates the expression for ∇,

$$\frac{\partial}{\partial x}\mathbf{a}^* + \frac{\partial}{\partial y}\mathbf{b}^* + \frac{\partial}{\partial z}\mathbf{c}^*,$$

changes into

$$\frac{\partial}{\partial \xi}\mathbf{p}^* + \frac{\partial}{\partial \eta}\mathbf{q}^* + \frac{\partial}{\partial \zeta}\mathbf{r}^*.$$

If x, y, z refer to a self reciprocal system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, then evidently

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k},$$

since the distinction from the reciprocal system disappears.

If a vector field $\mathbf{g}$ is given and is derivable from a scalar function f by means of the formula

$$\mathbf{g} = \nabla f,$$

then $-f$ is called the potential function of the vector field. The term originates from mechanics. If, for example, $\mathbf{g}$ represents the force experienced by a point mass situated at any given position of the field, we may associate with the point mass a definite "potential energy" or energy of

* "Nabla" is the expression for a harp-like musical instrument whose shape is represented by the symbol ∇.

position which, during a displacement of the mass $d\mathbf{r}$, undergoes a change:

$$-\mathbf{g} \cdot d\mathbf{r} = -df.$$

If the point mass is translated along a curve from a point A to another point P, then

$$f_A - f_P = \int_A^P -\mathbf{g} \cdot d\mathbf{r},$$

the total change in potential energy. As is evident from the left-hand side of the equation, it depends only on the positions of the points A and P and not on the path along which the mass was displaced. If the point A is held fixed while P is considered to vary, we note that an addition of $-f$ during the change in position gives the quantity of energy that must be contributed to the point mass in order to produce the displacement in question in the field of force, while a diminution of this amount, on the other hand, would give the quantity of energy that would be delivered up by the mass. Apart then from an arbitrary constant $-f$ measures the "potential energy," that is, the energy of position existing in the point mass, and is therefore called the potential function or simply the potential.

Applying the operator ∇ to a vector field $\mathbf{p}$ in such a way that the external product

$$\nabla\mathbf{p}$$

is taken, we derive a field of vectorial area. If u, v, w are the coefficients of $\mathbf{p}$ referred to the reciprocal system $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$, then the external product is

$$\nabla\mathbf{p} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{b}^*\mathbf{c}^* + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{c}^*\mathbf{a}^* + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{a}^*\mathbf{b}^*$$

or the vectorial product equals the complement of this, viz.

$$\nabla \times \mathbf{p} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\mathbf{a}}{\mathbf{abc}} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\mathbf{b}}{\mathbf{abc}} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\mathbf{c}}{\mathbf{abc}}.$$

This vectorial area, just as with the vector ∇f already discussed, must be independent of the initially selected system **a**, **b**, **c**, to which the variables x, y, z refer, as can easily be proved.

Finally, applying the operator ∇ to the field of vectorial areas **F** by taking the external product

$$\nabla \mathbf{F}$$

we again derive a scalar function. If p, q, r are the coefficients of the representation of **F** referred to **a**, **b**, **c**, then

$$\bar{\nabla} \mathbf{F} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z}.$$

Once again this scalar function, as we shall presently see, is independent of the unit vectors **a**, **b**, **c**.

In addition to the operator ∇ we propose to introduce its representation :

$$| \nabla = \frac{\partial}{\partial x}\frac{\mathbf{bc}}{\mathbf{abc}} + \frac{\partial}{\partial y}\frac{\mathbf{ca}}{\mathbf{abc}} + \frac{\partial}{\partial z}\frac{\mathbf{ab}}{\mathbf{abc}}.$$

For the same reasons as allow us to refer to ∇ as a vector, we may refer to $|\nabla$ as a vectorial area. Applied to a scalar function f, to a vector field **p**, and to a vectorial area **F**, the operator $|\nabla$ gives in the first case a field of vectorial areas, in the second a scalar function, and in the third a vector field,

$$| (f\nabla) = f\,|\,\nabla = \frac{\partial f}{\partial x}\frac{\mathbf{bc}}{\mathbf{abc}} + \frac{\partial f}{\partial y}\frac{\mathbf{ca}}{\mathbf{abc}} + \frac{\partial f}{\partial z}\frac{\mathbf{ab}}{\mathbf{abc}},$$

$$\nabla \cdot \mathbf{p} = \mathbf{p} \cdot \nabla = \mathbf{p}\,|\,\nabla = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} + \frac{\partial r}{\partial z},$$

$$\mathbf{F}\,|\,\nabla = \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)\mathbf{a} + \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\mathbf{b} + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\mathbf{c}.$$

Here p, q, r are the coefficients of **p** referred to **a**, **b**, **c** and u, v, w the coefficients of **F** referred to **bc**, **ca**, **ab**.

All three quantities, as we shall presently see, are independent of the unit vectors.

§ 8. THE TRANSFORMATION OF SURFACE INTO VOLUME INTEGRALS

Let the boundary of a body be supposed divided up in elementary vectorial areas $d\mathfrak{G}$ whose *sense* is so chosen that the interior of the body lies on the positive side. The body lies inside the region for which f, $\mathbf{p}$, and $\mathbf{F}$ are defined so that we may integrate over the boundary, first the product $fd\mathfrak{G}$, secondly the external product $\mathbf{p}d\mathfrak{G}$, and thirdly the external product $\mathbf{F}d\mathfrak{G}$.

The following three formulæ then apply:

1. $$\int fd\mathfrak{G} + \int f \mid \nabla d\tau = 0,$$

2. $$\int \mathbf{p}d\mathfrak{G} + \int \mathbf{p} \mid \nabla d\tau = 0,$$

3. $$\int \mathbf{F}d\mathfrak{G} + \int \mathbf{F} \mid \nabla d\tau = 0,$$

where $d\tau$ in the second term is a positive element of volume and the integration is to be extended throughout the volume of the body.

To prove these, imagine the body split up into portions but the various parts assembled in their original positions. Each integral may then be replaced by the sum of the integrals over the separate parts. For the volume integral this is self-evident. That it is true also for the surface integrals follows from the fact that each portion of the boundary of a part lying inside the body appears twice, as the common boundary of two parts. For the one part, however, the *sense* is opposite to that for the other part. Consequently the sum of the corresponding portions of the integrals vanish and there remain only those portions of the boundary of the parts which together form the original boundary of the body.

Let now one of the parts be of parallelepiped form, the sides of which are parallel to the co-ordinate planes and let the edges presumed small be of lengths Δx, Δy, Δz. Let one corner have co-ordinates x, y, z, the other corners being derived from it by increasing the appropriate co-ordinates by Δx, Δy, Δz.

Then for the sides parallel to the yz-plane

$$d\mathfrak{a} = dydz\mathbf{bc} \text{ and } d\mathfrak{a} = -dydz\mathbf{bc},$$

where the positive sign corresponds to the side with the smaller x co-ordinate. Hence for these two surfaces we obtain

$$\int f(x, y, z)dydz\mathbf{bc}$$

and

$$-\int f(x + \Delta x, y, z)dydz\mathbf{bc},$$

i.e. neglecting terms of higher order

$$-\frac{\partial f}{\partial x}\Delta x\Delta y\Delta z\mathbf{bc}.$$

The contributions of the sides parallel to the xz- and xy-planes are derived by cyclical substitution, so that neglecting terms of higher order all the terms together give the vectorial area:

$$-f \mid \nabla \Delta x\Delta y\Delta z\mathbf{abc}.$$

The volume integral $\int f \mid \nabla d\tau$ throughout the same portion provides

$$+f \mid \Delta\, \Delta x\Delta y\Delta z\mathbf{abc}$$

as far as terms of this order.

Together the contributions of both integrals amount to terms of higher order than $\Delta x\Delta y\Delta z$. Hence it follows that the sum of both integrals over the parallelepiped whose sides are parallel to the co-ordinate planes must vanish. For if the parallelepiped is split up into n^3 similar ones, each of them contributes a quantity which as n increases becomes small in comparison with $1/n^3$. Hence the total value cannot be different from zero. The proof for every such parallelepiped region involves also the justification of the formula for any arbitrarily bounded body. For if the interior of the body is supposed split up into a sufficient number of small parallelepipeds, those portions of parallelepipeds remaining at the boundary contribute merely an indefinitely small amount to the two integrals. For the threefold integral

this follows directly from the smallness of the total volume of these portions. For the surface integral it follows from the fact that for a sufficiently small portion

$$\int f d\mathfrak{G} = f \int d\mathfrak{G}$$

on neglecting terms of higher order. But as we have already remarked, for the surface of a body

$$\int d\mathfrak{G} = 0.$$

Consequently the total contribution of all these small portions is of the order of their volume, i.e. indefinitely small.

A proof quite analogous to that carried through for case 1 may be developed for cases 2 and 3. We need merely consider again a parallelepiped whose edges are of length $\Delta x \Delta y \Delta z$. In case 2, for the side perpendicular to the x-axis $\int \mathbf{p} d\mathfrak{G}$ contributes

$$\int \mathbf{p}(x, y, z) dx dy \mathbf{bc} - \int \mathbf{p}(x + \Delta x, y, z) dx dy \mathbf{bc},$$

that is to say, neglecting terms of higher order

$$- \frac{\partial \mathbf{p}}{\partial x} \Delta x \Delta y \Delta z \mathbf{bc} = - \left(\frac{\partial p}{\partial x} \mathbf{a} + \frac{\partial q}{\partial x} \mathbf{b} + \frac{\partial r}{\partial x} \mathbf{c} \right) \nabla x \nabla y \Delta z \mathbf{bc}$$

$$= - \frac{\partial p}{\partial x} \Delta x \Delta y \Delta z \mathbf{abc}.$$

Thus the whole surface of the parallelepiped provides, neglecting terms of higher order

$$- \mathbf{p} \mid \nabla \, \Delta x \Delta y \Delta z \mathbf{abc},$$

which again disappears when added to the contribution from the volume integral. In case 3 for the side parallel to the yz-plane, $\int \mathbf{F} d\mathfrak{G}$ contributes

$$\int \mathbf{F}(x, y, z) dx dy \mathbf{bc} - \int \mathbf{F}(x + \Delta x, y, z) dx dy \mathbf{bc},$$

that is, neglecting terms of higher order,

$$-\left(\frac{\partial u}{\partial x}\mathbf{bc} + \frac{\partial v}{\partial x}\mathbf{ca} + \frac{\partial w}{\partial x}\mathbf{ab}\right)\Delta x\Delta y\Delta z\mathbf{bc},$$

i.e.

$$\left(\frac{\partial v}{\partial x}\mathbf{c} - \frac{\partial w}{\partial x}\mathbf{b}\right)\Delta x\Delta y\Delta z\mathbf{abc}.$$

Hence the whole surface of the parallelpiped provides a term

$$-\mathbf{F} \mid \nabla\, \Delta x\Delta y\Delta z\mathbf{abc},$$

neglecting higher order terms, and this again disappears when added to the contribution from the volume integral.

The passage to an arbitrary body over whose surface and throughout whose volume the integrals are extended, is carried through exactly as in the first case, and we need not repeat it here.

From the three theorems we may assert the following conclusions, that the three quantities

$$f \mid \nabla, \quad \mathbf{p} \mid \nabla, \quad \mathbf{F} \mid \nabla$$

are independent of the choice of the unit vectors of reference. For a sufficiently small body of volume $\Delta\tau$ the three volume integrals are equal to

$$f \mid \nabla\Delta\tau, \quad \mathbf{p} \mid \nabla\Delta\tau, \quad \mathbf{F} \mid \nabla\Delta\tau,$$

neglecting higher order terms.

Hence as far as terms vanishingly small in comparison with $\Delta\tau$:

$$\mathbf{f} \mid \nabla = -\frac{1}{\Delta\tau}\int f d\mathfrak{a},$$

$$\mathbf{p} \mid \nabla = -\frac{1}{\Delta\tau}\int \mathbf{p} d\mathfrak{a},$$

$$\mathbf{F} \mid \nabla = -\frac{1}{\Delta\tau}\int \mathbf{F} d\mathfrak{a}.$$

Thus $f \mid \nabla$, $\mathbf{p} \mid \nabla$, and $\mathbf{F} \mid \nabla$ may be defined as the limiting values to which the right-hand sides approach as

$\Delta\tau$ diminishes. Since, however, the right-hand sides are independent of the choice of unit vectors, it follows that the same is true for the left-hand side.

In place of the vectorial area $f \mid \nabla$ we may also introduce its *representation*, the above defined vector ∇f. For $\mathbf{p} \mid \nabla$ we may also write the scalar product $\nabla \cdot \mathbf{p}$, and for $\mathbf{F} \mid \nabla$ we may write also the vectorial product of the two representations of $\mathbf{F}$ and $\mid \nabla$, that is $\mid \mathbf{F} \times \nabla$ or $- \nabla \times \mid \mathbf{F}$ This succeeds, however, in hiding the fact of the similarity in type of the three integral propositions.

§ 9. Applications of the Theorems of Transformation

Let a body be completely immersed in a heavy fluid at rest. The pressure on its surface is a scalar function of position which we designate by p. The force exerted by the fluid pressure on the element of surface $d\mathfrak{G}$ may be expressed by the representation of the vectorial area

$$p d\mathfrak{G}.$$

The resultant of the total fluid pressure is consequently equal to the representation of the vectorial area

$$\int p d\mathfrak{G},$$

where the integral extends over the surface of the body.

From the first of the three formulæ the resultant is consequently equal to the representation of

$$- \int p \mid \nabla d\tau,$$

that is, the resultant itself is equal to

$$- \int \nabla p d\tau.$$

If, in illustration the fluid be assumed incompressible of density ρ, the force of gravity being set equal to $g\mathbf{c}$, then the pressure at a depth z below the surface is :

$$\rho g z.$$

Hence the gradient of the pressure is

$$\nabla p = \rho g \mathbf{c}$$

and the total resultant thrust

$$-\int \rho g d\tau \mathbf{c},$$

i.e. the resultant is equal and opposite to the weight of displaced liquid. Formula (1) § 8 is completely represented, physically if we imagine the space occupied by the body entirely filled with liquid. For each element of the fluid $d\tau$, the pressure experienced on its surface balances the weight, which must therefore have the value

$$\nabla p d\tau.$$

Thus ∇p is the vector field for gravity, measured per unit volume.

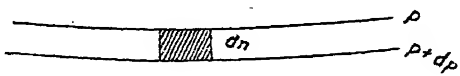


FIG. 26.

The simplest point of view is to imagine two surfaces in the liquid

$$p = \text{constant, and } p + dp = \text{constant,}$$

and between them a cylindrical element of height dn and cross-section $d\tau/dn$ (fig. 26). In order to balance the excess of pressure dp on the area $d\tau/dn$, the weight of the element must be in the direction of the normal from the surface p = constant to the surface $p + dp$ = constant and be of amount $dpd\tau/dn$, that is to say, it must equal the vector $\nabla pd\tau$.

Formula (2) may be interpreted physically in a different manner. Imagine a vector field determined by taking the product $\rho\mathbf{v}$ of the density and velocity at each point of a moving fluid in space, and suppose an arbitrary region is separated off. The vector $\rho\mathbf{v}$ is then at each instant, a function of position, that is to say, we need not limit the case to that of steady streaming where the vector is

independent of time. Every vector field $\mathbf{p}$, dependent in any arbitrary manner on time, provides the possibility for constructing such a physical picture of a moving fluid. At any particular instant we may even assume the density ρ to be an arbitrary positive scalar function. As will be evident from what follows the density will be found from the vector field $\mathbf{p}$ for all time, and when ρ is determined we immediately derive

$$\mathbf{v} = \mathbf{p}/\rho.$$

If as before the vectorial area $d\mathfrak{G}$ is an element of the surface with a sense so chosen that the interior of the space lies on its positive side, then the external product

$$\rho \mathbf{v} d\mathfrak{G} dt$$

represents the quantity of fluid which streams through $d\mathfrak{G}$ in the element of time dt, reckoned positive if entering the region and negative if leaving.

The integral

$$\int \rho \mathbf{v} d\mathfrak{G} dt,$$

extended over the whole boundary of the region, will therefore specify by its positive or negative sign whether more fluid enters or leaves the region and will give the quantity by which the fluid in the region has increased or diminished during the element of time dt. In each element of volume $d\tau$ the density changes during the time dt by $\frac{\partial \rho}{\partial t} dt$. The value of the integral may therefore also be expressed in the form

$$\int \frac{\partial \rho}{\partial t} d\tau dt;$$

hence

$$\int \frac{\partial \rho}{\partial t} d\tau = \int \rho \mathbf{v} d\mathfrak{G}.$$

Accordingly, formula (2), § 8, may be written

$$\int \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) d\tau = 0,$$

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and it follows, since the region is bounded in any arbitrary manner

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0.$$

Thus formula (2) expresses simply the fact of the balance of matter for the region considered. The change in amount present in the volume may be calculated, on the one hand, in terms of the fluid streaming through the boundary, and on the other hand, by the change in density in each element of space. The surface integral derived from the first calculation

$$\int \mathbf{p} d\mathfrak{G}, \; (\mathbf{p} = \rho \mathbf{v})$$

must consequently be equal to the volume integral

$$-\int \mathbf{p} \mid \nabla d\tau$$

derived from the second calculation, as formula (2) asserts.

The expression $\mathbf{p} \mid \nabla$, or what is the same $\nabla \cdot \mathbf{p}$, is called the "divergence" of the vector field $\mathbf{p}$. The term originates from the physical meaning of $\nabla \cdot \mathbf{v}$, where $\mathbf{v}$ represents the vector field of the velocity of a moving fluid. Since, as we have seen above,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v} = -\nabla \rho \cdot \mathbf{v} - \rho \nabla \cdot \mathbf{v},$$

we may also write

$$-\frac{1}{\rho} \frac{d\rho}{dt} = \nabla \cdot \mathbf{v},$$

where $\frac{d\rho}{dt}$ stands for $\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v}$, and measures the change of density of an element of fluid per unit of time, which in the element of time dt has acquired a displacement $\mathbf{v}dt$. For the variation of its density with time is composed of a change $\frac{\partial \rho}{\partial t} dt$ (x, y, and z remaining unaltered) and a change because of the alteration of x, y, z by amounts udt, vdt, wdt, that is

$$\frac{\partial \rho}{\partial x} udt + \frac{\partial \rho}{\partial y} vdt + \frac{\partial \rho}{\partial z} wdt = \nabla \rho \cdot \mathbf{v}dt.$$

The value of $\nabla \cdot \mathbf{v}$ thus furnishes the relative change of density per unit time in such a way that a positive value of $\nabla \cdot \mathbf{v}$ indicates a thinning, an expansion (divergence) of the fluid at the point in question. Hence $\nabla \cdot \mathbf{v}$ is called the "divergence" of the vector field $\mathbf{v}$, and the conception is extended to any arbitrary vector field $\mathbf{p}$, even when $\mathbf{p}$ is not a velocity, but has some other physical significance.

If the divergence of a vector field $\mathbf{p}$ is zero

$$\mathbf{p} \mid \nabla = \nabla \cdot \mathbf{p} = 0,$$

then from formula (2) it appears that for the boundary of any arbitrary body in the field

$$\int \mathbf{p} d\mathfrak{G} = 0.$$

Imagine once again a fluid in motion, of density ρ and velocity $\mathbf{v}$ such that

$$\mathbf{p} = \rho \mathbf{v},$$

then in consequence of the equation

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot \mathbf{p} = 0$$

the density at each point is independent of time. If now it is assumed to be 1 everywhere at any instant then it continues to have the value 1, and thus

$$\mathbf{p} = \mathbf{v}.$$

The vector field $\mathbf{p}$, whose divergence disappears everywhere, may be taken to represent the field of velocity vectors of a streaming incompressible fluid of density 1. The equation

$$\int \mathbf{p} d\mathfrak{G} = 0$$

then asserts that for the region over whose boundary the integral is taken at each instant as much fluid enters as leaves.

If we care we may take the vector field as fixed so that it does not vary with time. The streaming is then steady. Let the fluid enter into the region through any small portion

of the boundary, and follow the streaming element until it leaves the region at some other point of the boundary.

This part of the fluid we term a stream-filament. The strength of the current flowing in it, that is to say, the quantity of fluid passing any section of it per unit of time, is measured by the integral

$$\int \mathbf{p} d\mathfrak{G} = \epsilon$$

extended over that part of the boundary where the stream filament enters the region; we may, however, equally well calculate the current by taking the integral over any other arbitrary section of the stream filament and selecting $d\mathfrak{G}$ an element of the section in such a way that the current flows from the negative to the positive side. That $\int \mathbf{p} d\mathfrak{G}$ has the same value for every section of the stream filament follows directly from the physical meaning of the integral since it is evident that for an incompressible steadily moving fluid a constant quantity of fluid flows through per unit time; on the other hand, it is also clear mathematically if we consider the region between two sections and take the integral

$$\int \mathbf{p} d\mathfrak{G}$$

over the boundary of this region. According to formula (2), § 8, in consequence of the evanescence of the divergence of $\mathbf{p}$ this integral must vanish. Now on the surface of the stream filament itself $\mathbf{p} d\mathfrak{G}$ is everywhere zero since $\mathbf{p}$ is parallel to the vectorial area $d\mathfrak{G}$. Hence the two portions of the integral over the cross-sections of the filaments are equal and opposite, or what is in effect the same, they are equal if in both cases the element $d\mathfrak{G}$ is so selected that the current flows from the negative to the positive side.

Such a vector field $\mathbf{p}$, for which the divergence $\nabla \mathbf{p}$ vanishes, may consequently be supposed constituted of a large number of stream filaments of equal strength ϵ. To evaluate the integral

$$\int \mathbf{p} d\mathfrak{G}$$

for any portion of the boundary of any region it is merely necessary to calculate the number of stream filaments which cross, reckoning them as positive when they correspond to flow from the negative to the positive side of $d\mathfrak{G}$, and vice versa. The number of stream filaments multiplied by ϵ then provides the value of the integral. This striking method of evaluation was first applied to the magnetic field, where **p** represents the force experience at the point in question by a magnetic pole of unit strength. A magnetic field has consequently the property that its divergence at each point vanishes.

In order to give a physical interpretation to formula (3), § 8, also, let us again consider the case of a body immersed in a heavy fluid in order to evaluate the moment exerted upon it by the pressure of the fluid.

The pressure on the element $d\mathfrak{G}$ is the vector whose complement, as we have already remarked, equals

$$p\,d\mathfrak{G},$$

where p again represents the pressure. If $\mathbf{r}$ stands for the position vector of the element $d\mathfrak{G}$ then the moment of the force referred to the point O, from which the position vector is drawn, may be represented by the vectorial product of the position vector and the force, or what is in effect the same thing, the external product of the complement of $\mathbf{r}$ with $p\,d\mathfrak{G}$, the complement of the force.

Hence the total moment of the fluid pressures referred to O is given by the vector

$$\int \mathbf{R}\,p\,d\mathfrak{G},$$

if $\mathbf{R}$ is the complement of the position vector. Let us apply formula (3), § 8, to this integral by introducing $\mathbf{R}p$ for the vectorial area $\mathbf{F}$. The turning moment may then also be written as the volume integral:

$$-\int \mathbf{F} \mid \nabla d\tau.$$

The vector $\mathbf{F} \mid \nabla$ can now be separated into two portions corresponding to the two factors of $\mathbf{F}$. If p were independent of position we would only have the portion $p(\mathbf{R} \mid \nabla)$;

if **R** were independent of position we would merely have the portion $\mathbf{R}(p \mid \nabla)$ or $\mathbf{R} \mid (\nabla p)$. If p and **R** are both dependent on position, then

$$\mathbf{F} \mid \nabla = p(\mathbf{R} \mid \nabla) + \mathbf{R} \mid \nabla p,$$

or, as we may also write this,

$$\mathbf{F} \mid \nabla = p(\mathbf{r} \times \nabla) + \mathbf{r} \times \nabla p.$$

Now by direct expansion of

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)$$

we see at once that $\mathbf{r} \times \nabla$ is zero.

Hence

$$\mathbf{F} \mid \nabla = \mathbf{r} \times \nabla p,$$

and the turning moment assumes the form

$$-\int \mathbf{r} \times \nabla p d\tau.$$

Imagine the body replaced by fluid, then, as we have already found from equation (1), § 8, the weight of an element of fluid is given by $\nabla p d\tau$, so that $\mathbf{r} \times \nabla p d\tau$ is the moment of the weight referred to the point O. Thus formula (3), § 8, merely asserts that the pressure of the liquid on the surface has a moment equal and opposite to that of the weight of the fluid inside the boundary. It would appear that this is nothing more than the condition for the equilibrium of the fluid. If an arbitrary portion of the fluid is bounded off the forces operating on it must be in equilibrium, and these forces consist of the pressures on the elements of surface and of the weights of the elements of fluid. The pressure may be any scalar function of position which then fixes the gravitational field. Equilibrium is effected by having the sum of the moments of all these forces referred to any arbitrary point, zero.

§ 10. The Transformation of Line Integrals into Surface Integrals

To the three formulæ (1), (2), (3), § 8, there correspond three others in which the volume integral is replaced by one

over a portion of the bounding surface and the surface integral by one over the edge of that boundary.

Let $d\mathbf{r}$ represent an element of the edge and $d\mathfrak{G}$ an element of the bounding surface so selected that the *sense* of $d\mathfrak{G}$ is the same as would be obtained by describing the boundary edge in the direction $d\mathbf{r}$.

The following three formulæ apply:

$$(1^*) \int f d\mathbf{r} + \int (f \mid \nabla) d\mathfrak{G} = 0;$$

$$(2^*) \int \mathbf{p} d\mathbf{r} - \int (\mathbf{p} \mid \nabla) d\mathfrak{G} = 0;$$

where $\mathbf{p}$ is a vector everywhere parallel to the vectorial area $d\mathfrak{G}$

$$(3^*) \int \mathbf{F} d\mathbf{r} + \int (\mathbf{F} \mid \nabla) d\mathfrak{G} = 0.$$

The proofs are very similar to those for the previous three formulæ.

Imagine the surface divided into a large number of small elements and the boundary of each element described in the same sense as that of the original, then the line integrals in each of the three formulæ may be replaced by the sum of the integrals for the elements. For all the line boundaries common to two elements of the surface are described during the integration twice, in opposite directions, so that the corresponding contribution to the total integral is zero. There remain only those elements of the line boundaries which are also elements of the original line boundary, and they are described in the integration in the same sense as that of the original boundary.

Now let us suppose the elements of the surface so small that we may regard them as plane, and let us take the x- and y-axes parallel to the sides of a rectangle so that the line integral from the corner x, y extends to $x + \Delta x$, y; then to $x + \Delta x$, $y + \Delta y$; then to x, $y + \Delta y$; and back to x, y. Let the unit vectors be $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, which are self reciprocal. To fix our ideas consider formula (1*). The two sides parallel to the y-axis then furnish

$$\left(\int f(x + \Delta x, y) dy - \int f(x, y) dy\right) \mathbf{j},$$

that is, neglecting terms of higher order

$$\frac{\partial f}{\partial x}\Delta x\Delta y\mathbf{j},$$

and the two sides parallel to the x-axis

$$-\frac{\partial f}{\partial y}\Delta x\Delta y\mathbf{i}.$$

Regarding the rectangle as a vectorial area let it be represented by $\Delta\mathbf{G}$. Then:

$$\Delta\mathbf{G} = \Delta x\Delta y\mathbf{ij}.$$

The line integral

$$\int f d\mathbf{r} = \left(\frac{\partial f}{\partial x}\mathbf{j} - \frac{\partial f}{\partial y}\mathbf{i}\right)\Delta x\Delta y + \text{terms of higher order}$$

may, neglecting higher order terms, be written as the external product of the two vectorial areas:

$$\Delta\mathbf{G} \text{ and } \left(\frac{\partial f}{\partial x}\mathbf{jk} + \frac{\partial f}{\partial y}\mathbf{ki} + \frac{\partial f}{\partial z}\mathbf{ij}\right) = f \mid \nabla.$$

For in the external product

$$\Delta\mathbf{G}\,(f \mid \nabla)$$

we have

$$\begin{aligned}(\mathbf{ij})\,(\mathbf{jk}) &= \mathbf{j},\\ (\mathbf{ij})\,(\mathbf{ki}) &= -\,\mathbf{i},\\ (\mathbf{ij})\,(\mathbf{ij}) &= 0,\end{aligned}$$

so that we get

$$\left(\frac{\partial f}{\partial x}\mathbf{j} - \frac{\partial f}{\partial y}\mathbf{i}\right)\Delta x\Delta y.$$

Accordingly:

$$\int f d\mathbf{r} = \Delta\mathbf{G}\,(f \mid \nabla) + \text{terms of higher order.}$$

Since $\Delta\mathbf{G}$ and $f \mid \nabla$ are independent of the system of co-ordinates so also is $\Delta\mathbf{G}(f \mid \nabla)$.

Apart from a remainder which may be made arbitrarily small by taking $\Delta\mathbf{G}$ sufficiently small, the whole line integral

may be expressed in terms of the sum of the external products $\Delta\mathbf{G}(f \mid \nabla)$, and therefore the whole line integral is the limit value approached by the sum of the quantities $\Delta\mathbf{G}(f \mid \nabla)$ when $\Delta\mathbf{G}$ becomes sufficiently small.

Thus

$$\int f d\mathbf{r} = \int d\mathbf{G}(f \mid \nabla),$$

or

$$\int f d\mathbf{r} + \int (f \mid \nabla) d\mathbf{G} = 0.$$

The splitting up into rectangles may be conceived as occurring by drawing over the surface two orthogonal systems of curves. The non-rectangular elements remaining at the boundary, together give a quantity merely of the order of their total area which, by taking a sufficient number of curves of the orthogonal systems, may be made an indefinitely small fraction of the given surface.

Formula (2*) may be established in the same manner. The line integral taken over the sides of the rectangular vectorial area $\Delta\mathbf{G}$ gives a quantity

$$\left(\frac{\partial \mathbf{p}}{\partial x}\mathbf{j} - \frac{\partial \mathbf{p}}{\partial y}\mathbf{i}\right)\Delta x \Delta y,$$

on neglecting higher order terms.

Since $\mathbf{p}$ is parallel to $\Delta\mathbf{G}$, it must be derivable numerically from $\mathbf{i}$ and $\mathbf{j}$, thus:

$$\mathbf{p} = u\mathbf{i} + v\mathbf{j}.$$

Hence

$$\frac{\partial \mathbf{p}}{\partial x}\mathbf{j} - \frac{\partial \mathbf{p}}{\partial y}\mathbf{i} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\mathbf{ij} = (\mathbf{p} \mid \nabla)\,\mathbf{ij},$$

and consequently neglecting terms of higher order for the boundary of the rectangle

$$\left(\frac{\partial \mathbf{p}}{\partial x}\mathbf{j} - \frac{\partial \mathbf{p}}{\partial y}\mathbf{i}\right)\Delta x \Delta y = (\mathbf{p} \mid \nabla)\Delta\mathbf{G},$$

where the right-hand side is once again independent of the

choice of co-ordinate system. For the same reasons as before we derive for the whole line integral

$$\int \mathbf{p} d\mathbf{r} = \int (\mathbf{p} \mid \nabla) d\mathfrak{G}.$$

In case (3^*) for the edges of $\Delta\mathfrak{G}$, neglecting again terms of higher order, we find

$$\left(\frac{\partial \mathbf{F}}{\partial x}\mathbf{j} - \frac{\partial \mathbf{F}}{\partial y}\mathbf{i}\right)\Delta x \Delta y,$$

which once again may be written in a form independent of the co-ordinate system

$$-(\mathbf{F} \mid \nabla)\Delta\mathfrak{G}.$$

For the whole line integral we obtain once again in the same manner

$$-\int (\mathbf{F} \mid \nabla) d\mathfrak{G},$$

and hence formula (3^*)

$$(3^*) \qquad \int \mathbf{F} d\mathbf{r} + \int (\mathbf{F} \mid \nabla) d\mathfrak{G} = 0.$$

If the vector which is the representation of the vectorial area $\mathbf{F}$ is called $\mathbf{f}$, then the formula (3^*) may also be written

$$\int \mathbf{f} \,.\, d\mathbf{r} + \int (\mathbf{f} \times \nabla) d\mathfrak{G} = 0,$$

$(\mathbf{f} \times \nabla)\, d\mathfrak{G}$ being positive or negative according as the vector $\mathbf{f} \times \nabla$ is directed to the positive or to the negative side of $d\mathfrak{G}$, and its numerical value being equal to the volume described when the vectorial area $d\mathfrak{G}$ is displaced by an amount specified vectorially by $\mathbf{f} \times \nabla$.

If $\mathbf{f} \times \nabla = 0$ over the whole surface to which formula (3^*) applies, then:

$$\int \mathbf{f} \,.\, d\mathbf{r} = 0.$$

If A and B are two points on the line boundary, then the two portions of this integral, corresponding respectively to the path along the boundary from A to P, and from P to A,

will be equal and opposite, or expressed otherwise the value of the integral

$$\int_A^P \mathbf{f} \, . \, d\mathbf{r}$$

is independent of whether the path is taken along the one or the other side of the boundary. Any other path, in fact, from A to P gives the same value for the integral provided it can be derived from this one by continuous variation in a region for which $\mathbf{F} \mid \nabla$ is zero. For the two paths together would then define the boundary of a surface for which $\mathbf{F} \mid \nabla$ vanishes. If now the point A is maintained fixed, while P is allowed to vary, then the integral

$$\int_A^P \mathbf{f} \, . \, d\mathbf{r}$$

defines a scalar function f of the position of P. By displacing the point P through $d\mathbf{r}$ the function alters by an amount $\mathbf{f} \, . \, d\mathbf{r}$. Thus the vector $\mathbf{f}$ is the gradient of the scalar function. Conversely, if the vector $\mathbf{f}$ be the gradient of a scalar function f, then

$$\mathbf{f} \times \nabla = 0.$$

For then

$$\int \mathbf{f} \, . \, d\mathbf{r} = \int df = 0,$$

where the integral is taken over the boundary of a region for which the gradient is defined.

From this it follows, according to our formula, that

$$\int (\mathbf{f} \times \nabla) d\mathfrak{G} = 0,$$

or since the boundary is arbitrary

$$\mathbf{f} \times \nabla = 0.$$

This may be obtained directly from the fact that

$$\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \times \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) = 0,$$

since the order of differentiation may be changed.

The property of the vector field

$$\mathbf{f} \times \nabla = 0$$

is therefore not merely the sufficient but also the necessary condition that the vector $\mathbf{f}$ should be the gradient of a scalar function, or what amounts to the same thing, that

$$\mathbf{f} \cdot d\mathbf{r}$$

should be a complete differential.

When the integral

$$\int \mathbf{f} \cdot d\mathbf{r},$$

taken over a closed curve, assumes a value different from zero it follows that if any surface whatever is taken with its edges on the given curve, there must lie a point or points on this surface where $\mathbf{f} \times \nabla$ is not zero. Those portions of the surface for which $\mathbf{f} \times \nabla$ is zero contribute nothing to the surface integral but this is not the case for those for which $\mathbf{f} \times \nabla$ is different from zero. These alone determine the value of the surface integral, and consequently from (3*) of the line integral also. By an alteration of the bounding curve in such a manner that those portions of the surface are removed or added for which $\mathbf{f} \times \nabla$ vanishes the value of the line integral will be unaffected. In other words, the curve in space may be continuously altered without changing the integral provided the region described by the curve in varying is such that $\mathbf{f} \times \nabla$ vanishes everywhere within it. When by such variation the curve can collapse into a point without leaving the region for which $\mathbf{f} \times \nabla$ is zero, the line integral must have the value zero. A region which has the property that every closed curve drawn in it can be continuously reduced to a point in this way is termed *simply connected*. If a vector field $\mathbf{f}$ is defined in a simply connected region for which $\mathbf{f} \times \nabla$ is zero, then the integral

$$\int \mathbf{f} \cdot d\mathbf{r}$$

taken over any closed curve in the region must vanish.

A region is multiply connected when closed curves in it cannot be reduced to a point without leaving the region, for example the space enclosed by a finger ring. A closed curve which circumscribes the finger once inside the ring

may indeed be continuously varied into every other curve which does the same but it cannot be arbitrarily reduced. If now a vector field $\mathbf{f}$ were defined which in the region of the ring possessed the property

$$\mathbf{f} \times \nabla = 0,$$

then the case could arise where

$$\int \mathbf{f} \,.\, d\mathbf{r}$$

integrated over a closed curve could have a value different from zero. For all closed curves which encircle the finger once inside the ring in the same sense, the value of the integral must be the same, while if it encircles it n-times the value is n-times this. The scalar function f, which is derived by the integration from a fixed point A to a variable point P is then multiple valued. For the same point P the value of f may be altered an arbitrary positive or negative integral number of times a certain constant by encircling the ring an arbitrary number of times in the one or the opposite sense before arriving at P. Such a region is termed doubly connected. The region outside the ring is likewise doubly connected. Every closed curve external to the ring, but which does not thread it, may be reduced to a point, while every closed curve threading the ring once may be reduced to any other which does the same. If we imagine an electromotive force applied inside the ring, so that an electric current is set up, a magnetic field $\mathbf{f}$ will originate which, as is shown in the theory of electricity, possesses the property that outside the ring where there is no electrical flow,

$$\mathbf{f} \times \nabla = 0;$$

inside the ring, however, where there exists an electric current,

$$(\nabla \times \mathbf{f}) d\mathfrak{a},$$

measured in the customary electrical units, is equal to 4π times the current through $d\mathfrak{a}$, reckoned positive when it flows towards the positive side of $d\mathfrak{a}$.

Now imagine a surface cutting the ring at some position and let us integrate

$$\int \mathbf{f} \cdot d\mathbf{r}$$

around the boundary of this section. From formula (3*) this line integral is equal to the surface integral

$$\int (\nabla \times \mathbf{f}) d\mathfrak{a},$$

and its sign indicates at once whether the electric current flows from the positive to the negative side of the surface or vice versa, while its numerical value divided by 4π measures the quantity of electricity passing per unit time. In other words, the constant

$$\frac{1}{4\pi} \int \mathbf{f} \cdot d\mathbf{r}$$

is a measure of the current flowing in the ring. The form of the curve is immaterial, always provided it threads the ring.

The vector

$$\nabla \times \mathbf{f}$$

is called the *spin* or the *rotation* or the *rotor* of the field $\mathbf{f}$ at the point in question. The term originates from a consideration of a fluid in motion. As we shall show later, if $\mathbf{f}$ represents the velocity vector field, then

$$\tfrac{1}{2} \nabla \times \mathbf{f}$$

represents the rotational velocity at the point. The direction of this vector

$$\tfrac{1}{2} \nabla \times \mathbf{f}$$

gives the direction of the axis of rotation in such a way that the rotation is that of a right-handed screw moving in the direction of the vector. The length of the vector represents the magnitude of the rotational speed.

Thus the vector field $\mathbf{f}$, when its rotation is not zero, that is, when $\mathbf{f}$ is not the gradient of a scalar function, leads to a second vector field :

$$\mathbf{g} = \nabla \times \mathbf{f}.$$

The divergence of this second field (see § 9) is zero. It is:

$$\nabla \cdot \mathbf{g} = \nabla \cdot (\nabla \times \mathbf{f}).$$

Now let the three factors on the right-hand side be cyclically interchanged:

$$\nabla \cdot (\nabla \times \mathbf{f}) = \nabla \cdot (\mathbf{f} \times \nabla),$$

but

$$\mathbf{f} \times \nabla = - \nabla \times \mathbf{f},$$

consequently

$$\nabla \cdot (\nabla \times \mathbf{f}) = - \nabla \cdot (\nabla \times \mathbf{f}),$$

which therefore vanishes, analogously with the formula stating that the parallelepiped formed from three vectors vanishes when two of the vectors become equal. The derived vector field $\mathbf{g}$ is apparently then of the type discussed above (§ 9), where $\mathbf{g}$ may be conceived as the velocity of an incompressible fluid in steady motion and the vector field may be separated out into stream filaments all of which have the same current strength.

§ 11. Introduction of Curvilinear Co-ordinates

The quantities

$$f \mid \nabla, \quad \mathbf{p} \mid \nabla, \quad \mathbf{F} \mid \nabla,$$

which have been derived from the scalar function f of position, from the vector field $\mathbf{p}$, and from the field of vectorial area $\mathbf{F}$, might be defined as the limiting value assumed by the ratio of the integrals

$$\int f \mid \nabla d\tau = - \int f d\mathfrak{G};$$

$$\int \mathbf{p} \mid \nabla d\tau = - \int p d\mathfrak{G};$$

$$\int \mathbf{F} \mid \nabla d\tau = - \int \mathbf{F} d\mathfrak{G};$$

to the volume $\Delta\tau$ when the volume $\Delta\tau$ through which the integration is to be extended shrinks to a point. From this we can derive the simple expressions given above for $f \mid \nabla$, $\mathbf{p} \mid \nabla$, $\mathbf{F} \mid \nabla$. Let a system of co-ordinates be determined

also be written in the form

$$\nabla = \frac{\partial}{\partial \xi}\mathbf{e}^* + \frac{\partial}{\partial \eta}\mathbf{f}^* + \frac{\partial}{\partial \zeta}\mathbf{g}^*.\dagger$$

In the same way:

$$\mid \nabla = \frac{\partial}{\partial \xi} \mid \mathbf{e}^* + \frac{\partial}{\partial \eta} \mid \mathbf{f}^* + \frac{\partial}{\partial \zeta} \mid \mathbf{g}^*$$

$$= \frac{1}{\mathbf{efg}}\left(\frac{\partial}{\partial \xi}\mathbf{fg} + \frac{\partial}{\partial \eta}\mathbf{ge} + \frac{\partial}{\partial \zeta}\mathbf{ef}\right).$$

But this form of the operator must equally well be capable of being applied to the vector field

$$\mathbf{p} = p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c}$$

or to a field of vectorial area

$$\mathbf{F} = f_1\mathbf{bc} + f_2\mathbf{ca} + f_3\mathbf{ab}.$$

For the external multiplication by ∇ and $\mid \nabla$ arises in the formation of gradients $\nabla\mathbf{p}$ and their *representations*, e.g.

$$\nabla\mathbf{p} = \nabla p_1\mathbf{a} + \nabla p_2\mathbf{b} + \nabla p_3\mathbf{c}.$$

If the vector $\mathbf{p}$ is referred to the vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{F}$ to the vectorial areas $\mathbf{fg}$, $\mathbf{ge}$, $\mathbf{ef}$, then we have merely to note in applying the operator ∇ or $\mid \nabla$ that $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{fg}$, $\mathbf{ge}$, $\mathbf{ef}$ are not constant vectors and vectorial areas but depend on position and must therefore also be involved in the differentiation.

In the three formulæ

$$\int f d\mathbf{G} + \int f \mid \nabla d\tau = 0,$$

$$\int \mathbf{p} d\mathbf{G} + \int \mathbf{p} \mid \nabla d\tau = 0,$$

$$\int \mathbf{F} d\mathbf{G} + \int \mathbf{F} \mid \nabla d\tau = 0,$$

the volume integrals may also be expressed in terms of curvilinear co-ordinates ξ, η, ζ. It is merely necessary to

† In particular $\nabla\xi = \mathbf{e}^*$; $\nabla\eta = \mathbf{f}^*$, $\nabla\zeta = \mathbf{g}^*$.

replace the volume element $d\tau$ by the external product of the three vectors

$$\frac{\partial \mathbf{r}}{\partial \xi} d\xi, \ \frac{\partial \mathbf{r}}{\partial \eta} d\eta, \ \frac{\partial \mathbf{r}}{\partial \zeta} d\zeta,$$

i.e.

$$\mathbf{efg} \, d\xi \, d\eta \, d\zeta,$$

and for $| \nabla$ the operator

$$\frac{\mathbf{fg}}{\mathbf{efg}} \frac{\partial}{\partial \xi} + \frac{\mathbf{ge}}{\mathbf{efg}} \frac{\partial}{\partial \eta} + \frac{\mathbf{ef}}{\mathbf{efg}} \frac{\partial}{\partial \zeta},$$

that is, for $| \nabla d\tau$ we have to substitute

$$\left(\mathbf{fg} \frac{\partial}{\partial \xi} + \mathbf{ge} \frac{\partial}{\partial \eta} + \mathbf{ef} \frac{\partial}{\partial \zeta}\right) d\xi d\eta d\zeta.$$

If a region is bounded off by three pairs of surfaces ξ and $\xi + \Delta\xi$, η and $\eta + \Delta\eta$, ζ and $\zeta + \Delta\zeta$, then, neglecting terms of higher order:

$$- \int d\mathfrak{G} = \left(\frac{\partial \mathbf{fg}}{\partial \xi} + \frac{\partial \mathbf{ge}}{\partial \eta} + \frac{\partial \mathbf{ef}}{\partial \zeta}\right) \Delta\xi \Delta\eta \Delta\zeta.$$

Since for the two surfaces ξ and $\xi + \Delta\xi$ we have

$$\int d\mathfrak{G} = \int [(\mathbf{fg})_{\xi} - (\mathbf{fg})_{\xi + \Delta\xi}] \, d\eta d\zeta$$

or

$$- \int d\mathfrak{G} = \int \frac{\partial \mathbf{fg}}{\partial \xi} d\eta d\zeta d\xi + \text{terms of higher order}$$

$$= \frac{\partial \mathbf{fg}}{\partial \xi} \Delta\xi \Delta\eta \Delta\zeta + \text{terms of higher order.}$$

Now since $\int d\mathfrak{G}$ is zero when integrated over the whole boundary of any arbitrary volume, it follows that

$$\frac{\partial \mathbf{fg}}{\partial \xi} + \frac{\partial \mathbf{ge}}{\partial \eta} + \frac{\partial \mathbf{ef}}{\partial \zeta} = 0,$$

for if it were different from zero then for sufficiently small values of $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ the quantity $\int d\mathfrak{G}$ would also be different from zero.

With the aid of this equation we may throw the quantities $f \mid \nabla$, $\mathbf{p} \mid \nabla$, $\mathbf{F} \mid \nabla$, into yet another form. We add to each of these the quantity of zero value,

$$\frac{1}{\mathbf{efg}}\left(\frac{\partial \mathbf{fg}}{\partial \xi} + \frac{\partial \mathbf{ge}}{\partial \eta} + \frac{\partial \mathbf{ef}}{\partial \zeta}\right)$$

multiplied by f, $\mathbf{p}$ or $\mathbf{F}$, and obtain

$$f \mid \nabla = \frac{1}{\mathbf{efg}}\left(\mathbf{fg}\frac{\partial f}{\partial \xi} + \mathbf{ge}\frac{\partial f}{\partial \eta} + \mathbf{ef}\frac{\partial f}{\partial \zeta}\right)$$

$$+ \frac{1}{\mathbf{efg}}\left(f\frac{\partial \mathbf{fg}}{\partial \xi} + f\frac{\partial \mathbf{ge}}{\partial \eta} + f\frac{\partial \mathbf{ef}}{\partial \zeta}\right)$$

or

$$f \mid \nabla = \frac{1}{\mathbf{efg}}\left(\frac{\partial f\mathbf{fg}}{\partial \xi} + \frac{\partial f\mathbf{ge}}{\partial \eta} + \frac{\partial f\mathbf{ef}}{\partial \zeta}\right)$$

with the analogous formulæ

$$\mathbf{p} \mid \nabla = \frac{1}{\mathbf{efg}}\left(\frac{\partial \mathbf{pfg}}{\partial \xi} + \frac{\partial \mathbf{pge}}{\partial \eta} + \frac{\partial \mathbf{pef}}{\partial \zeta}\right),$$

$$\mathbf{F} \mid \nabla = \frac{1}{\mathbf{efg}}\left(\frac{\partial \mathbf{Ffg}}{\partial \xi} + \frac{\partial \mathbf{Fge}}{\partial \eta} + \frac{\partial \mathbf{Fef}}{\partial \zeta}\right);$$

by $\mathbf{Ffg}$, etc., is understood the vector formed from the external product of the vectorial areas $\mathbf{F}$ and $\mathbf{fg}$, or written otherwise

$$\mathbf{Ffg} = \mid \mathbf{F} \times \mid \mathbf{fg} = (\mathbf{efg}) \mid \mathbf{F} \times \mathbf{e}^*.$$

As we have already remarked, the *representation* of $f \mid \nabla$ is termed the gradient of the scalar function f, and $\mathbf{p} \mid \nabla$ the divergence of the vector field $\mathbf{p}$, while $- \mathbf{F} \mid \nabla$ the rotation of the vector field $\mid \mathbf{F}$. This then indicates the procedure for expressing gradient, divergence, and rotation in curvilinear co-ordinates also.

As an example consider the polar co-ordinates in space, r, θ, ϕ in terms of which we can write:

$\mathbf{r} = r \sin\theta \cos\phi\mathbf{i} + r\sin\theta\sin\phi\mathbf{j} + r\cos\theta\mathbf{k}$. From this we derive

$$\mathbf{e} = \frac{\partial \mathbf{r}}{\partial r} = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k},$$

$$\mathbf{f} = \frac{\partial \mathbf{r}}{\partial \theta} = r\cos\theta\cos\phi\mathbf{i} + r\cos\theta\sin\phi\mathbf{j} - r\sin\theta\mathbf{k},$$

$$\mathbf{g} = \frac{\partial \mathbf{r}}{\partial \phi} = -r\sin\theta\sin\phi\mathbf{i} + r\sin\theta\cos\phi\mathbf{j},$$

$$d\mathbf{r} = dr\mathbf{e} + d\theta\mathbf{f} + d\phi\mathbf{g},$$

$$d\mathbf{r}\,.\,d\mathbf{r} = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$

For

$$\mathbf{e}\,.\,\mathbf{e} = 1,\ \mathbf{f}\,.\,\mathbf{f} = r^2,\ \mathbf{g}\,.\,\mathbf{g} = r^2\sin^2\theta,$$

$$\mathbf{f}\,.\,\mathbf{g} = \mathbf{g}\,.\,\mathbf{e} = \mathbf{e}\,.\,\mathbf{f} = 0$$

and thus

$$\mathbf{efg} = r^2\sin^2\theta,$$

$$\mathbf{e} = (\mathbf{e}\,.\,\mathbf{e})\mathbf{e}^* + (\mathbf{e}\,.\,\mathbf{f})\mathbf{f}^* + (\mathbf{e}\,.\,\mathbf{g})\mathbf{g}^* = \mathbf{e}^*,$$

$$\mathbf{f} = (\mathbf{f}\,.\,\mathbf{e})\mathbf{e}^* + (\mathbf{f}\,.\,\mathbf{f})\mathbf{f}^* + (\mathbf{f}\,.\,\mathbf{g})\mathbf{g}^* = r^2\mathbf{f}^*,$$

$$\mathbf{g} = (\mathbf{g}\,.\,\mathbf{e})\mathbf{e}^* + (\mathbf{g}\,.\,\mathbf{f})\mathbf{f}^* + (\mathbf{g}\,.\,\mathbf{g})\mathbf{g}^* = r^2\sin^2\theta\mathbf{g}^*,$$

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}^* + \frac{\partial}{\partial \theta}\mathbf{f}^* + \frac{\partial}{\partial \phi}\mathbf{g}^*$$

$$= \frac{\partial}{\partial r}\mathbf{e} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\mathbf{f} + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\mathbf{g},$$

$$|\,\nabla = \left(\frac{\partial}{\partial r}\mathbf{fg} + \frac{\partial}{\partial \theta}\mathbf{ge} + \frac{\partial}{\partial \phi}\mathbf{ef}\right)\frac{1}{r^2\sin\theta}.$$

The gradient of a scalar function f is in terms of polar space co-ordinates equal to:

$$\dot{\nabla} f = \frac{\partial f}{\partial r}\mathbf{e} + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\mathbf{f} + \frac{1}{r^2\sin^2\theta}\frac{\partial f}{\partial \phi}\mathbf{g}.$$

If $$\mathbf{p} = u\mathbf{e} + v\mathbf{f} + w\mathbf{g}$$

the divergence of the vector field $\mathbf{p}$ is most easily obtained in the form

$$\mathbf{p}\,|\,\nabla = \frac{1}{\mathbf{efg}}\left(\frac{\partial \mathbf{pfg}}{\partial \xi} + \frac{\partial \mathbf{pge}}{\partial \eta} + \frac{\partial \mathbf{pef}}{\partial \zeta}\right).$$

Now

$$\mathbf{pfg} = u\mathbf{efg},\ \mathbf{pge} = v\mathbf{efg},\ \mathbf{pef} = w\mathbf{efg},$$

and therefore if we write $\mathbf{efg} = \omega$, then

$$\mathbf{p} \mid \nabla = \frac{1}{\omega}\left(\frac{\partial \omega u}{\partial \xi} + \frac{\partial \omega v}{\partial \eta} + \frac{\partial \omega w}{\partial \zeta}\right)$$

$$= \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{\partial \omega}{\partial \zeta} + \frac{u \partial \omega}{\omega\, \partial \xi} + \frac{v}{\omega}\frac{\partial \omega}{\partial \eta} + \frac{w}{\omega}\frac{\partial \omega}{\partial \zeta}.$$

If the operator $\mid \nabla$ is applied directly to $\mathbf{p}$ we find

$$\mathbf{p} \mid \nabla = \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{\partial \omega}{\partial \zeta} + u(\mathbf{e} \mid \nabla) + v(\mathbf{f} \mid \nabla) + w(\mathbf{g} \mid \nabla),$$

and since the result must be the same as in the first case we conclude that

$$\frac{u}{\omega}\frac{\partial \omega}{\partial \xi} + \frac{v}{\omega}\frac{\partial \omega}{\partial \eta} + \frac{w}{\omega}\frac{\partial \omega}{\partial \zeta}$$

$$= u(\mathbf{e} \mid \nabla) + v(\mathbf{f} \mid \nabla) + w(\mathbf{g} \mid \nabla).$$

But the quantities u, v, w are arbitrary functions of ξ, η, ζ, hence

$$\frac{1}{\omega}\frac{\partial \omega}{\partial \xi} = \mathbf{e} \mid \nabla,$$

$$\frac{1}{\omega}\frac{\partial \omega}{\partial \eta} = \mathbf{f} \mid \nabla,$$

$$\frac{1}{\omega}\frac{\partial \omega}{\partial \zeta} = \mathbf{g} \mid \nabla,$$

that is, the gradient of $\log \omega$ is equal to

$$\nabla \log \omega = (\mathbf{e} \mid \nabla)\mathbf{e}^* + (\mathbf{f} \mid \nabla)\mathbf{f}^* + (\mathbf{g} \mid \nabla)\mathbf{g}^*.$$

This may also be seen directly by differentiating the external product

$$\omega = \mathbf{efg}.$$

$$\frac{\partial \omega}{\partial \xi} = \frac{\partial \mathbf{e}}{\partial \xi}\mathbf{fg} + \frac{\partial \mathbf{f}}{\partial \xi}\mathbf{ge} + \frac{\partial \mathbf{g}}{\partial \xi}\mathbf{ef}.$$

Now

$$\mathbf{f} = \frac{\partial \mathbf{r}}{\partial \eta}$$

$$\frac{\partial \mathbf{f}}{\partial \xi} = \frac{\partial^2 \mathbf{r}}{\partial \xi \partial \eta} = \frac{\partial \mathbf{e}}{\partial \eta},$$

$$\mathbf{g} = \frac{\partial \mathbf{r}}{\partial \zeta},$$

$$\frac{\partial \mathbf{g}}{\partial \eta} = \frac{\partial^2 \mathbf{r}}{\partial \eta \partial \zeta} = \frac{\partial \mathbf{e}}{\partial \zeta},$$

and therefore

$$\frac{\partial \omega}{\partial \xi} = \frac{\partial \mathbf{e}}{\partial \xi}\mathbf{fg} + \frac{\partial \mathbf{e}}{\partial \eta}\mathbf{ge} + \frac{\partial \mathbf{e}}{\partial \zeta}\mathbf{ef}$$

so that by dividing by ω,

$$\frac{\partial \log \omega}{\partial \xi} = \mathbf{e} \mid \nabla$$

with analogous expressions for η and ζ.

If $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ are mutually perpendicular then as a rule mathematical writers prefer to express the vectors not in terms of $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, but three vectors parallel to these and of length unity, viz.:

$$\frac{\mathbf{e}}{\sqrt{\mathbf{e}\,.\,\mathbf{e}}}, \frac{\mathbf{f}}{\sqrt{\mathbf{f}\,.\,\mathbf{f}}}, \frac{\mathbf{g}}{\sqrt{\mathbf{g}\,.\,\mathbf{g}}}.$$

Instead of the coefficients u, v, w of the vector

$$\mathbf{p} = u\mathbf{e} + v\mathbf{f} + w\mathbf{g},$$

$\bar{u}$, $\bar{v}$, and $\bar{w}$ are introduced, so that

$$\mathbf{p} = \bar{u}\frac{\mathbf{e}}{\sqrt{\mathbf{e}\,.\,\mathbf{e}}} + \bar{v}\frac{\mathbf{f}}{\sqrt{\mathbf{f}\,.\,\mathbf{f}}} + \bar{w}\frac{\mathbf{g}}{\sqrt{\mathbf{g}\,.\,\mathbf{g}}}$$

$$\bar{u} = u\sqrt{\mathbf{e}\,.\,\mathbf{e}},\ \bar{v} = v\sqrt{\mathbf{f}\,.\,\mathbf{f}},\ \bar{w} = w\sqrt{\mathbf{g}\,.\,\mathbf{g}}.$$

With polar co-ordinates in space, for example, the vectors of reference would be:

$$\mathbf{e}, \frac{\mathbf{f}}{r}, \frac{\mathbf{g}}{r \sin \theta}.$$

The coefficients of the gradient

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}^* + \frac{\partial f}{\partial \theta}\mathbf{f}^* + \frac{\partial f}{\partial \phi}\mathbf{g}^*$$

$$= \frac{\partial f}{\partial r}\mathbf{e} + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\mathbf{f} + \frac{1}{r^2 \sin^2\theta}\frac{\partial f}{\partial \phi}\mathbf{g}$$

$$= \frac{\partial f}{\partial r}\mathbf{e} + \frac{1}{r}\frac{\partial f}{\partial \theta}\frac{\mathbf{f}}{r} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\frac{\mathbf{g}}{r\sin\theta}$$

then become

$$\frac{\partial f}{\partial r},\ \frac{1}{r}\frac{\partial f}{\partial \theta},\ \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi},$$

and the divergence

$$\mathbf{p} \mid \nabla = \frac{1}{r^2 \sin\theta}\left(\frac{\partial r^2 \sin\theta u}{\partial r} + \frac{\partial r^2 \sin\theta \bar{v}}{\partial \theta} + \frac{\partial r^2 \sin\theta \bar{\bar{w}}}{\partial \phi}\right)$$

may be written

$$\frac{1}{r^2 \sin\theta}\left(\frac{\partial r^2 \sin\theta u}{\partial r} + \frac{\partial r \sin\theta \bar{v}}{\partial \theta} + \frac{\partial r\bar{w}}{\partial \phi}\right).$$

The symmetry of the formulæ is of course affected.

The divergence of the gradient of a scalar function is written $\nabla f \mid \nabla$ and becomes

$$\frac{1}{\omega}\left(\frac{\partial \nabla f\,\mathbf{fg}}{\partial \xi} + \frac{\partial \nabla f\,\mathbf{ge}}{\partial \eta} + \frac{\partial \nabla f\,\mathbf{ef}}{\partial \zeta}\right)$$

where

$$\nabla f = \frac{\partial f}{\partial \xi}\mathbf{e}^* + \frac{\partial f}{\partial \eta}\mathbf{f}^* + \frac{\partial f}{\partial \zeta}\mathbf{g}^*.$$

Here $\mathbf{e}^*$, $\mathbf{f}^*$, $\mathbf{g}^*$ are to be expressed in terms of $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$,

$$\mathbf{e}^* = (\mathbf{e}^* . \mathbf{e}^*)\mathbf{e} + (\mathbf{e}^* . \mathbf{f}^*)\mathbf{f} + (\mathbf{e}^* . \mathbf{g}^*)\mathbf{g}, \text{ etc.}$$

Then:

$$\nabla f \mid \nabla = \frac{1}{\omega}\left(\frac{\partial \omega u}{\partial \xi} + \frac{\partial \omega v}{\partial \eta} + \frac{\partial \omega w}{\partial \zeta}\right).$$

In illustration, if $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ are mutually perpendicular then

$$\begin{aligned} \mathbf{e} &= (\mathbf{e}\,.\,\mathbf{e})\mathbf{e}^* \\ \mathbf{f} &= (\mathbf{f}\,.\,\mathbf{f})\mathbf{f}^* \\ \mathbf{g} &= (\mathbf{g}\,.\,\mathbf{g})\mathbf{g}^* \end{aligned}$$

and consequently

$$\omega u = \frac{\omega}{\mathbf{e}\,.\,\mathbf{e}} \frac{\partial f}{\partial \xi}$$

$$\omega v = \frac{\omega}{\mathbf{f}\,.\,\mathbf{f}} \frac{\partial f}{\partial \eta}$$

$$\omega w = \frac{\omega}{\mathbf{g}\,.\,\mathbf{g}} \frac{\partial f}{\partial \zeta}$$

For polar co-ordinates in space, for example, we would have

$$\nabla f \mid \nabla = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 \sin\theta \frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(r \sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial \phi^2}\right];$$

for cylindrical co-ordinates

$$\begin{aligned} \mathbf{r} &= r\cos\phi\mathbf{i} + r\sin\phi\mathbf{j} + z\mathbf{k} \\ \mathbf{e} &= \cos\phi\mathbf{i} + \sin\phi\mathbf{j}, \\ \mathbf{f} &= -r\sin\phi\mathbf{i} + r\cos\phi\mathbf{j}, \\ \mathbf{g} &= \mathbf{k}, \qquad \mathbf{efg} = r, \\ \mathbf{e}\,.\,\mathbf{e} &= 1,\ \mathbf{f}\,.\,\mathbf{f} = r^2,\ \mathbf{g}\,.\,\mathbf{g} = 1, \\ \mathbf{f}\,.\,\mathbf{g} &= \mathbf{g}\,.\,\mathbf{e} = \mathbf{e}\,.\,\mathbf{f} = 0, \end{aligned}$$

so that

$$\nabla f \mid \nabla = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.$$

The rotation of the vector field $\mid \mathbf{F}$ is calculated from the coefficients of $\mathbf{F}$ with reference to $\mathbf{fg}$, $\mathbf{ge}$, $\mathbf{ef}$:

$$\mathbf{F} = f_1\mathbf{fg} + f_2\mathbf{ge} + f_3\mathbf{ef}$$

$$\mid \mathbf{F} = \omega f_1\mathbf{e}^* + \omega f_2\mathbf{f}^* + \omega f_3\mathbf{g}^*$$

$$\mathbf{F} \mid \nabla = \frac{1}{\omega}\left[\frac{\partial}{\partial \xi}(\omega f_3\mathbf{f} - \omega f_2\mathbf{g}) + \frac{\partial}{\partial \eta}(\omega f_1\mathbf{g} - \omega f_3\mathbf{e}) + \frac{\partial}{\partial \zeta}(\omega f_2\mathbf{e} - \omega f_1\mathbf{f})\right].$$

§ 12. Rules for the Operator ∇

The operator ∇ satisfies a number of definite laws which we propose to set out here in detail:

(1) $$\nabla(f + g) = \nabla f + \nabla g,$$

where instead of the scalar functions f and g two vectors or two vectorial areas may be substituted and the product with ∇ is to be regarded as the external product. If f and g are scalar functions we obtain a vectorial equation, if they are vectors an equation involving vectorial areas is derived, and if they are vectorial areas the equation is a scalar relation. The rule is also correct if in place of ∇ its representation $|\nabla$ is substituted, and also when the factors on both sides are interchanged:

$$(f + g) \mid \nabla = f \mid \nabla + g \mid \nabla.$$

It also remains true if the representation of the products is taken on both sides. Thus, for example, from

$$\nabla(\mathbf{f} + \mathbf{g}) = \nabla\mathbf{f} + \nabla\mathbf{g}$$

it follows by taking the representations that

$$\nabla \times (\mathbf{f} + \mathbf{g}) = \nabla \times \mathbf{f} + \nabla \times \mathbf{g}.$$

Further:

(2) $$\nabla(fg) = (\nabla f)g + f(\nabla g).$$

This formula is still true if a vector or a vectorial area is taken instead of the scalar function g, and the external products selected in the products of the vectors or vectorial areas with the operator ∇ regarded as a vector. It is, however, important to note that the formula is no longer correct if for f a vector $\mathbf{f}$ is taken while g remains a scalar function. The correct formula in that case would run:

$$\nabla(\mathbf{f}g) = (\nabla\mathbf{f})g - \mathbf{f}(\nabla g),$$

which can be derived directly from

$$\nabla(f\mathbf{g}) = (\nabla f)\mathbf{g} + f(\nabla\mathbf{g})$$

if in this equation $\mathbf{f}$ is replaced by g and $\mathbf{g}$ by f. It then transforms into

$$\nabla(g\mathbf{f}) = (\nabla g)\mathbf{f} + g(\nabla\mathbf{f}).$$

On the left-hand side $g\mathbf{f}$ may be changed into $\mathbf{f}g$, and on the right-hand side $g(\nabla\mathbf{f})$ into $(\nabla\mathbf{f})g$, but the sign of $(\nabla g)\mathbf{f}$ is altered when the two factors are interchanged. The minus sign may also be understood from the fact that in the second term of the right-hand side of

$$\nabla(\mathbf{f}g) = (\nabla\mathbf{f})g - \mathbf{f}(\nabla g)$$

the order of the vectors ∇ and $\mathbf{f}$ is interchanged, compensation being obtained by the minus sign.

Also when two vectors $\mathbf{f}$ and $\mathbf{g}$ are introduced in place of f and g the formula remains correct; but here also a minus sign must be taken on the right-hand side:

$$\nabla(\mathbf{fg}) = (\Delta\mathbf{f})\,\mathbf{g} - \mathbf{f}(\nabla\mathbf{g}).$$

Of the four equations (2) which we have found in all, the original one where f and g are scalar functions is a vectorial equation; the second where g is replaced by a vector is an equation in vectorial areas. Finally the remaining two, where a vectorial area is taken instead of g, and where two vectors are inserted for f and g, are scalar equations.

The equation in vectorial areas

$$\nabla(f\mathbf{g}) = (\nabla f)\mathbf{g} + f(\nabla\mathbf{g}),$$

by using the *representations* by introducing the vectorial product, may also be written

$$\nabla\times(f\mathbf{g}) = (\nabla f)\times\mathbf{g} + f(\nabla\times\mathbf{g}).$$

Likewise the two scalar equations

$$\nabla(f\mathfrak{G}) = (\nabla f)\mathfrak{G} + f(\nabla\mathfrak{G})$$

and

$$\nabla(\mathbf{fg}) = (\nabla\mathbf{f})\mathbf{g} - \mathbf{f}(\nabla\mathbf{g}),$$

by introducing the representation of $\mathbf{G}$ (expressed by the symbol $\mathbf{g}$) and the scalar product, may be written in the form

$$\Delta \cdot (f\mathbf{g}) = (\nabla f) \cdot \mathbf{g} + f(\nabla \cdot \mathbf{g}),$$

and

$$\nabla \cdot (\mathbf{f} \times \mathbf{g}) = (\nabla \times \mathbf{f}) \cdot \mathbf{g} - \mathbf{f} \cdot (\nabla \times \mathbf{g}),$$

which does not however bring out the analogy between all the four equations so clearly.

In place of the operator ∇ treated as a vector, we may introduce its representation, the operator $|\nabla$ treated as a vectorial area. We then obtain likewise an equation for the product of the scalar functions f and g, and three equations derived from it, in which a vector or a vectorial area is inserted for one of the factors, and one equation in which both f and g are replaced by a vectorial area. Essentially, however, these equations contain nothing new, as can easily be shown.

A third property of the operator ∇ is obtained by external multiplication of the two operators ∇ and $|\nabla$. Since the one operator is to be treated as a vector the other as vectorial area, its external product becomes a scalar operator:

$$\nabla | \nabla = \nabla \cdot \nabla.$$

Applied to a scalar function f, it gives the divergence of the gradient of f. It has been customary to represent this operator by the Greek letter delta:

$$\Delta.$$

Referred to a self-reciprocal system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we have:

$$\Delta = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)$$
$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Referred to an arbitrary system of fixed or variable vectors, if the position vector $\mathbf{r}$ is

$$\mathbf{r} = \xi\mathbf{e} + \eta\mathbf{f} + \zeta\mathbf{g},$$

as has already been found, we have

$$\Delta f = \nabla f \mid \nabla = \frac{1}{\omega}\left(\frac{\partial \omega u}{\xi\partial} + \frac{\partial \omega v}{\partial \eta} + \frac{\partial \omega w}{\partial \zeta}\right),$$

where u, v, w, were defined by equation

$$\nabla f = \frac{\partial f}{\partial \xi}\mathbf{e}^* + \frac{\partial f}{\partial \eta}\mathbf{f}^* + \frac{\partial f}{\partial \zeta}\mathbf{g}^* = u\mathbf{e} + v\mathbf{f} + w\mathbf{g}.$$

The expression for the scalar operator in the variables ξ, η, ζ, is obtained by omitting the letter f where it occurs in u, v, w, in the expression for Δf.

By means of the operator

$$\Delta = \nabla \mid \nabla = \nabla \cdot \nabla,$$

applied to any function f of position, a new function of position Δf is derived.

We shall shortly consider the converse question whether if an arbitrary function of position is given as the value of ∇f, a scalar function f is determinable.

If the operator Δ is applied to a vector field $\mathbf{s}$, it must be noticed that $\nabla\mathbf{s}$ is not to be set equal to

$$(\nabla\mathbf{s}) \mid \nabla.$$

$\nabla\mathbf{s}$ is a vectorial area, and $\mid \nabla$ is also to be treated as a vectorial area. Following the analogy of the formula derived above

$$(\mathbf{pq}) \mid \mathbf{p} = (\mathbf{p} \times \mathbf{q}) \times \mathbf{p} = (\mathbf{p} \mid \mathbf{p})\mathbf{q} - (\mathbf{q} \mid \mathbf{p})\mathbf{p},$$

we get in this case

$$(\nabla\mathbf{s}) \mid \nabla = (\nabla \mid \nabla)\mathbf{s} - (\mathbf{s} \mid \nabla)\nabla$$

or

$$\begin{aligned}\Delta\mathbf{s} &= (\nabla \mid \nabla)\mathbf{s} \\ &= (\nabla\mathbf{s}) \mid \nabla + \nabla(\mathbf{s} \mid \nabla).\end{aligned}$$

Applied to the vector $\mathbf{s}$ then, the operator $\Delta = \nabla \mid \nabla$ gives rise to the sum of two vector fields both of which are independent of the system of co-ordinates. The one is constructed by taking the external product of the two vectorial

areas $\nabla \mathbf{s}$ and $| \nabla$, and may also be written as a vectorial product

$$(\nabla \times \mathbf{s}) \times \nabla,$$

while the other is the gradient of the divergence $\mathbf{s} \mid \nabla$ of the vector field $\mathbf{s}$. The case where the operator Δ is applied to a vectorial area may be reduced to that of a vector by introducing the complement. Since Δ is a scalar, the complement of $\Delta\mathrm{F}$ is equal to the result of operating on the complement of F.

§ 13. Application to Gravitational Potential

The gravitational potential of matter distributed throughout space with a certain finite density ρ, may be expressed in the form of a volume integral,

$$-\int \frac{\rho}{r} d\tau,$$

where $d\tau$ is the element of volume, ρ the density at the point in question, and r the distance of this point from the point P at which the potential is to be evaluated. In this the gravitation constant is taken as 1. The integral is to be extended to all elements of space at which matter exists. We will assume that it is limited to a finite portion of space. Let the integral

$$\int \frac{\rho}{r} d\tau$$

be set equal to f, which is therefore a scalar function of position, and the gravitational potential at the point P will be $-f$. If the point P at which we imagine the unit mass concentrated is moved a distance $d\mathbf{r}$, then the change in potential is equal to the scalar product

$$-\nabla f \,.\, d\mathbf{r}.$$

The sign of this expression indicates when it is positive that work is expended in order to cause the displacement

against the attractive force, and when it is negative that work is gained during the translation.

The field of force is therefore given by:

$$\nabla f.$$

Inserting the value for f we find

$$\nabla f = \int \rho \nabla \left(\frac{1}{r}\right) d\tau = - \int \rho \frac{\mathbf{r}}{r^3}\, d\tau,$$

where $\mathbf{r}$ represents the vector leading from the material particle $\rho d\tau$ to the point considered. For the gradient ∇f is composed of the sum of the gradients of each separate element

$$\frac{\rho}{r}\, d\tau,$$

and in order to obtain the gradient of the latter, since $\rho d\tau$ is constant, it is merely necessary to construct

$$\nabla \frac{1}{r} = - \frac{1}{r^2} \nabla r.$$

But ∇r has the direction of $\mathbf{r}$ and the length unity and is therefore equal to $\frac{\mathbf{r}}{\mathrm{r}}$. The field of force is given by means of this formula both for points inside and for points outside the matter.

Let us now integrate the vector ∇f over the boundary of an arbitrary volume; by changing the order of integration, we find:

$$\int \nabla f d\mathfrak{G} = - \int \rho \left\{ \int \frac{\mathbf{r} d\mathfrak{G}}{r^3} \right\} d\tau.$$

The inner integral is to be carried through for every point of the attractive matter. Now $\frac{1}{3}\, \mathbf{r} d\mathfrak{G}$ is equal to the volume of a cone whose vertex is at the particle of matter and whose base is $d\mathfrak{G}$, reckoned negative or positive according as the particle lies on the positive or on the negative side of $d\mathfrak{G}$.

Thus $\frac{1}{3}\frac{\mathbf{r}d\mathfrak{G}}{r^3}$ is equal to volume of the cone reduced to unit dimensions, taken positive if when viewed from the particle the negative side of $d\mathfrak{G}$ is seen, that is the outward turned side of $d\mathfrak{G}$. Now if the particle lies outside the space over which the integral

$$\int\frac{\mathbf{r}d\mathfrak{G}}{r^3}$$

is extended, then each indefinitely narrow cone proceeding outwards from it either does not meet the boundary of the space considered at all or in as many elements $d\mathfrak{G}$ of the one type as of the other, so that their total contribution will be zero. In this case, therefore, the integral will be zero. If, on the other hand, the particle lies inside the volume, then each of these cones will meet the boundary in one element $d\mathfrak{G}$ to be reckoned negative, or in addition in as many more positive as negative elements $d\mathfrak{G}$.

Accordingly the total value of the integral

$$\frac{1}{3}\int\frac{\mathbf{r}d\mathfrak{G}}{r^3}$$

in this case is equal to the volume of the sphere of unit radius taken negatively

$$\frac{1}{3}\int\frac{\mathbf{r}d\mathfrak{G}}{r^3} = -\frac{4}{3}\pi.$$

Inserting this value for the inner integral, we get

$$\int\nabla f d\mathfrak{G} = 4\pi\int\rho d\tau,$$

where on the right only those material particles enter into the calculation which lie inside the region over whose boundary the integral on the left-hand side is taken.

In other words,

$$\frac{1}{4\pi}\int\nabla f d\mathfrak{G}$$

is equal to the quantity of matter present within the region over whose boundary the integration is effected.

If the integral is transformed by means of formula (3), § 8, into a volume integral then we also have

$$-\frac{1}{4\pi}\int \Delta f d\tau = \int \rho d\tau,$$

where both integrals are taken over the same arbitrary region.

Hence we must have:

$$-\frac{\Delta f}{4\pi} = \rho.$$

For if at any point $-\dfrac{\Delta f}{4\pi}$ were different from ρ, the region of integration could be so restricted that the integrals of the two quantities would also be different.

Wherever $\rho = 0$, there must $\Delta f = 0$ also.

The scalar function

$$f = \int \frac{\rho}{r} d\tau$$

consequently, by deriving the divergence of the vector field obtained from its gradient, leads to a scalar function which differs from the density ρ merely by a constant factor -4π. This physical picture expresses in a striking manner the result already derived mathematically that Δf is a scalar function of position. But it contains more than this. It shows us, in fact, that if conversely Δf is a given function of position, a scalar function f can then be found. It is merely necessary to imagine matter distributed throughout space with density ρ assumed equal to

$$\rho = -\frac{1}{4\pi}\Delta f.$$

It is not necessary in this that ρ be positive. Negative values of ρ would correspond to matter of such a nature, that the force between it and matter of positive density would not be attractive but repulsive. From ρ, a function of position, the scalar function f is then constructed:

$$f = \int \frac{\rho}{r} d\tau.$$

The physical picture gives no indication, of course, whether other functions of position $\bar{f}$ may exist for which $\Delta\bar{f}$ is the same function of position as Δf. If a scalar product $\mathbf{r}\,.\,\mathbf{a}$ of the position vector $\mathbf{r}$ with an arbitrary constant vector $\mathbf{a}$ is added to f, then merely the constant vector $\mathbf{a}$ would be added to the field of force ∇f, and consequently nothing would be affected in $\Delta f = \nabla f \mid \nabla$, as is also obvious physically. Apart from this, however, are there also other types of scalar function $\bar{f}$?

To investigate this question let us construct the difference

$$u = \bar{f} - f$$

between two scalar functions both of which are presumed to lead to the same ρ. Then $\nabla u \mid \nabla$ must vanish over the whole region. The question is now, what can be concluded as regards the quantity u?

Consider the vector field $u\nabla u$. The divergence of this vector field, $u\nabla u \mid \nabla$ or, as we may alsow rite it, $\nabla \mid (u\nabla u)$ or $\nabla(u \mid \nabla u)$, is, from what has been done before,

$$\begin{aligned} u\nabla u \mid \nabla &= \nabla(u \mid \nabla u) \\ &= \nabla u \mid \nabla u + u(\nabla \mid \nabla u). \end{aligned}$$

Now
$$\nabla \mid \nabla u = \nabla u \mid \nabla = 0$$
so that
$$u\nabla u \mid \nabla = \nabla u \mid \nabla u.$$

From formula (2), § 8, by introducing the vector $u\nabla u$ in place of $\mathbf{p}$ we derive for the volume integral of $\nabla u \mid \nabla u$:

$$\int \nabla u \mid \nabla u d\tau = \int (u\nabla u) \mid \nabla d\tau = - \int u\nabla u d\mathfrak{G}.$$

Taking the surface integral over the boundary of a sphere of very large radius l about a point O as centre, and assuming that f and $\bar{f}$ and therefore that u become small of the order of $1/l$ as l increases, then the numerical value of ∇u becomes of the order of $1/l^2$ and that of $u\nabla u$ of the order of $1/l^3$. Since, however, the surface of the sphere is of the order l^2, the integral is of the order $1/l$, that is, it becomes indefinitely small with increasing l.

From this it is clear that

$$\nabla u \mid \nabla u$$

must be zero over the whole region. For it can never be negative being the square of the numerical value of ∇u. If at any point it differed from zero, the volume integral

$$\int \nabla u \mid \nabla u d\tau$$

extended over a sufficiently large sphere could not become arbitrarily small.

From $$\nabla u \mid \nabla u = 0$$
it follows that
$$\nabla u = 0,$$
and hence
$$du = \nabla u \,.\, d\mathbf{r} = 0$$
i.e. $$u = \text{constant}.$$

If it is required that the function f sought for shall be of the order $1/l$ at great distances l, then there is one and only one such function for which Δf is a prescribed function of position. In other words, with this limitation the passage from an arbitrary function of position f to the position function Δf is uniquely reversible and given by

$$f = -\frac{1}{4\pi}\int \frac{\Delta f}{r} d\tau.$$

§ 14. Green's Theorem

If f is a scalar function and $\mathbf{p}$ a vector field then, as we have already found,

$$\begin{aligned} f\mathbf{p} \mid \nabla &= \nabla \,.\, f\mathbf{p} \\ &= \nabla f \,.\, \mathbf{p} + f\nabla \,.\, \mathbf{p}, \end{aligned}$$

so that on using the transformation theorem (2), § 8,

$$\int f\mathbf{p} d\mathfrak{G} + \int \nabla f \,.\, \mathbf{p} d\tau + \int f\nabla \,.\, \mathbf{p} d\tau = 0.$$

If for $\mathbf{p}$ we put the gradient of a scalar function g, then:

$$\int f\nabla g d\mathfrak{G} + \int \nabla f \,.\, \nabla g d\tau + \int f\Delta g d\tau = 0.$$

Interchanging f and g in this equation and subtracting the new equation from the original one, we find :

$$\int(f\nabla g - g\nabla f)d\mathfrak{a} + \int(f\Delta g - g\Delta f)d\tau = 0.$$

This is termed Green's Theorem. We propose to use it by inserting in it a function f which throughout the whole region of integration satisfies the condition :

$$\Delta f = 0.$$

We then have

$$\int(f\nabla g - g\nabla f)d\mathfrak{a} + \int f\Delta g d\tau = 0,$$

where g is arbitrary.

The problem now proposed is to calculate the value of f at any point P inside the region of integration when f and ∇f are given on the boundary.

To this end suppose P enclosed in an exceedingly small space of volume ϵ and let us choose Δg so that inside this small space it has the constant value $-\ 4\pi\rho$ while it vanishes at all other points of the region. The integral

$$\int f\Delta g d\tau$$

merely requires to be extended over the small volume and is equal to

$$-\ 4\pi\rho\int f d\tau = -\ 4\pi\rho\epsilon\bar{f},$$

where $\bar{f}$ is the mean value of f in the small region.

If it be assumed that f is a continuous function of position, then

$$\int f d\tau = \bar{f}\epsilon,$$

and the mean value $\bar{f}$ differs by as little as we please from f_P, the actual value at the point P, if ϵ be assumed small enough.

Now for g we may write :

$$\int\frac{\rho d\tau}{r}.$$

The small region about P can now be made to shrink, while

at the same time ρ is allowed so to increase that the relation

$$\rho\epsilon = 1$$

remains continuously satisfied.

Then

$$\int f \Delta g d\tau$$

transforms into $\quad - 4\pi f_P$

and g becomes $\frac{1}{r}$,

where r is the distance of the point P; accordingly

$$\int \left[f \nabla \left(\frac{1}{r} \right) - \frac{1}{r} \nabla f \right] d\mathfrak{G} - 4\pi f_P = 0.$$

The value of the function f at the point P may be expressed as an integral over the boundary of the original region of integration, capable of being evaluated as soon as f and ∇f are given over the boundary.

If g were put equal to

$$\frac{1}{r} + g_1,$$

where g_1 is a scalar function which satisfies the condition

$$\Delta g_1 = 0$$

everywhere within the field of integration, then

$$\int (f \nabla g - g \nabla f) d\mathfrak{G}$$

would have the same value as for $g = \frac{1}{r}$, and just as before the value of f_P would be calculable by means of the integral

$$f_P = \frac{1}{4\pi} \int (f \nabla g - g \nabla f) d\mathfrak{G}.$$

It is to be noted that outside the region of integration Δg_1 could assume any arbitrary value, provided always $\Delta g_1 = 0$ inside the region.

If the values of Δg_1 outside the region of integration can be so chosen that along its boundary g is constant, then the calculation of f_P is much simplified, for it reduces to

$$f_P = \frac{1}{4\pi}\int f\nabla g d\mathfrak{G},$$

and the values of f alone, and no longer ∇f also, suffice to evaluate f_P. For the second part of the integral becomes

$$-\frac{g}{4\pi}\int \nabla f d\mathrm{G},$$

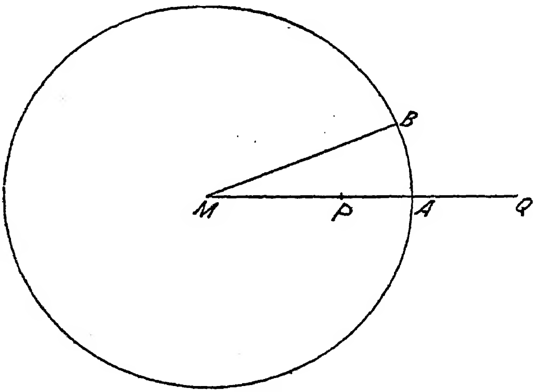

FIG. 27.

and according to formula (2), § 8,

$$-\int \nabla f d\mathfrak{G} = \int \Delta f d\tau,$$

which vanishes on account of the condition

$$\Delta f = 0.$$

In illustration suppose it is required to evaluate a function f at a point P inside a sphere of radius a from a knowledge of the values on the boundary of the sphere; we derive the function g in the following manner. Let the vector leading from the centre M in the direction of P to a point A on the

surface of the sphere be **a**, and $\lambda\mathbf{a}$ where $\lambda < 1$ be the vector from the centre M to P, then the vector $\frac{1}{\lambda}\mathbf{a}$ passes out through the sphere to a point Q (fig. 27). Further, let **b** be the vector from M to a point B on the sphere, then

$$\overline{PB}^2 = (\mathbf{b} - \lambda\mathbf{a}) \,.\, (\mathbf{b} - \lambda\mathbf{a}) = \mathbf{a}\,.\,\mathbf{a} - 2\lambda\mathbf{a}\,.\,\mathbf{b} + \lambda^2\mathbf{a}\,.\,\mathbf{a}$$

$$\overline{QB}^2 = (\mathbf{b} - \frac{1}{\lambda}\mathbf{a}) \,.\, (\mathbf{b} - \frac{1}{\lambda}) = \mathbf{a}\,.\,\mathbf{a} - \frac{2}{\lambda}\mathbf{a}\,.\,\mathbf{b} + \frac{1}{\lambda^2}\,\mathbf{a}\,.\,\mathbf{a},$$

since $\mathbf{b}\,.\,\mathbf{b} = \mathbf{a}\,.\,\mathbf{a}.$

Hence $\overline{PB}^2 = \lambda^2\,\overline{QB}^2.$

If we set

$$g = \frac{1}{r} - \frac{1}{\lambda r},$$

where r is the distance of P and r_1 the distance of Q, then g will vanish on the surface of the sphere. ∇g represents the field of force of an attractive mass 1 at P and a repelling mass $\frac{1}{\lambda}$ at Q. Δg will vanish everywhere except at the two points themselves, and $-\Delta\frac{1}{\lambda r_1}$ will vanish everywhere inside the sphere.

Accordingly:

$$f_P = \frac{1}{4\pi}\int f \nabla g\, dG.$$

§ 15. THE RELATION CONNECTING A VECTOR FIELD WITH ITS SPIN

In the foregoing we have considered the transition from an arbitrary vector field **f** to the vector field which we have termed the spin or rotor of **f**, viz.:

$$\mathbf{g} = \nabla \times \mathbf{f}.$$

We propose now to investigate the converse problem

whether to a given vector field **g**, there can be found a vector field **f** such that:

$$\mathbf{g} = \nabla \times \mathbf{f}.$$

For this purpose let us form

$$\begin{aligned}\mathbf{g} \times \nabla &= (\nabla \times \mathbf{f}) \times \nabla \\ &= (\nabla \mid \nabla)\mathbf{f} - \nabla(\mathbf{f} \mid \nabla)\end{aligned}$$

and associate with the required vector field **f** the additional condition, in the first instance, that its divergence

$$\mathbf{f} \mid \nabla$$

shall be zero.

Then we have:

$$\mathbf{g} \times \nabla = (\nabla \mid \nabla)\mathbf{f}.$$

The previously derived scalar equation

$$f = -\frac{1}{4\pi}\int \frac{(\nabla \mid \nabla)f}{r}\,d\tau$$

may now be extended at once to the vector equation

$$\mathbf{f} = -\frac{1}{4\pi}\int \frac{(\nabla \mid \nabla)\mathbf{f}}{r}\,d\tau,$$

since it holds for every coefficient of the vector if the latter is derived numerically from three fixed vectors. Inserting in this the expression $\mathbf{g} \times \nabla$ in place of $(\nabla \mid \nabla)\mathbf{f}$, we find

$$\mathbf{f} = \frac{1}{4\pi}\int \frac{\nabla \times \mathbf{g}}{r}\,d\tau.$$

If the vector field **f** does not satisfy the condition that the divergence $\mathbf{f} \mid \nabla$ vanishes, but if this divergence is known, then we can determine a scalar function u such that if the gradient of u is added to **f**, the divergence of the field $\bar{\mathbf{f}}$ so originated

$$\bar{\mathbf{f}} = \mathbf{f} + \nabla u$$

vanishes,

$$(\mathbf{f} + \nabla u) \mid \nabla = 0,$$

or

$$\nabla u \mid \nabla = -\mathbf{f} \mid \nabla;$$

thus

$$u = \frac{1}{4\pi}\int\frac{\mathbf{f} \mid \nabla}{r}\,d\tau.$$

Then, as we have already shown, $\bar{\mathbf{f}}$ can be found. For we have

$$\begin{aligned} \mathbf{g} &= \nabla \times (\bar{\mathbf{f}} - \nabla u) \\ &= \nabla \times \bar{\mathbf{f}}, \end{aligned}$$

and therefore

$$\bar{\mathbf{f}} = \frac{1}{4\pi}\int\frac{\nabla \times \mathbf{g}}{r}\,d\tau$$

and

$$\mathbf{f} = \frac{1}{4\pi}\int\frac{\nabla \times \mathbf{g}}{r}\,d\tau - \nabla u.$$

In other words, if in a vector field $\mathbf{f}$ the rotor $\mathbf{g} = \nabla \times \mathbf{f}$ and the divergence $\mathbf{f} \mid \nabla$ is given, then $\mathbf{f}$ is calculable by the above method.

We have already mentioned that if $\mathbf{f}$ is the velocity vector of an incompressible fluid, the rotor represents twice the angular rotation. In this case the divergence of $\mathbf{f}$ is zero and consequently:

$$\mathbf{f} = \frac{1}{4\pi}\int\frac{\nabla \times \mathbf{g}}{r}\,d\tau.$$

The rotor field which is derived from the velocity field by the operation $\nabla \times \mathbf{f}$ determines also, conversely, the velocity field, and in fact $\mathbf{f}$ is derived in a similar manner from $\nabla \times \mathbf{g}$ the rotor of the rotor just as a scalar function from the divergence of the field of its gradient.

If the divergence of the vector field $\mathbf{f}$ is not zero then in deriving $\mathbf{f}$ from the rotor of the rotor another gradient enters which is determined from the divergence of $\mathbf{f}$.

§ 16. Scalar Potential, Vector Potential, and Vectorial Area Potential

Besides the integral

$$\int\frac{\rho\,d\tau}{r},$$

by means of which, as we have just seen, the gravitational potential of a distribution of matter in space is expressed, two other integrals occur in many branches of physics which are of similar form, with this difference, that in place of the scalar function ρ there is a vector $\mathbf{p}$ or a vectorial area $\mathbf{P}$ dependent on the position of the element $d\tau$. It is worth while dealing with all three integrals together:

$$f = \int \frac{\rho d\tau}{r}, \quad \mathbf{f} = \int \frac{\mathbf{p} d\tau}{r}, \quad \mathbf{F} = \int \frac{\mathbf{P} d\tau}{r}.$$

In all three r stands for the distance of the element $d\tau$ from a point A, and all three are functions of the position of A, the first being a scalar function, the second a vector, and the third a vectorial area. In the three integrals r only depends on the position of A, the other quantities ρ, $\mathbf{p}$, $\mathbf{P}$ depending only on the position of the volume element $d\tau$, not on the position of A. From the three integrals f, $\mathbf{f}$, $\mathbf{F}$ we determine the three new integrals:

$$f \mid \nabla, \quad \mathbf{f} \mid \nabla, \quad \mathbf{F} \mid \nabla;$$

$$f \mid \nabla = \int \rho \left(\frac{1}{r} \mid \nabla \right) d\tau,$$

$$\mathbf{f} \mid \nabla = \int \mathbf{p} \left(\frac{1}{r} \mid \nabla \right) d\tau,$$

$$\mathbf{F} \mid \nabla = \int \mathbf{P} \left(\frac{1}{r} \mid \nabla \right) d\tau,$$

where for $\frac{1}{r} \mid \nabla$, the complement of

$$\frac{\mathbf{r}}{r^3}$$

may be written, if $\mathbf{r}$ is the vector leading from the point A to the element $d\tau$. $f \mid \nabla$ is the complement of the force $\mathbf{k}$:

$$\mathbf{k} = \int \frac{\rho}{r^2} \frac{\mathbf{r}}{r} d\tau,$$

with which the unit particle at A is attracted by the mass distributed in space (as before the gravitation constant is

taken equal to unity). For $\mathbf{f} \mid \nabla$ and $\mathbf{F} \mid \nabla$ also we propose to treat two examples having a physical interpretation. We have

$$\mathbf{p}\left(\frac{1}{r} \mid \nabla\right) = \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}$$

and

$$\mathbf{P}\left(\frac{1}{r} \mid \nabla\right) = \frac{\mathbf{P} \mid \mathbf{r}}{r^3}.$$

If the representation of $\mathbf{P}$ is $\mathbf{p}$, then we may write $\mathbf{p} \times \mathbf{r}$ for $\mathbf{P} \mid \mathbf{r}$, and accordingly:

$$\mathbf{f} \mid \nabla = \int \frac{\mathbf{p} \cdot \mathbf{r}}{r^3}\, d\tau,$$

$$\mathbf{F} \mid \nabla = \int \frac{\mathbf{p} \times \mathbf{r}}{r^3}\, d\tau.$$

If $\mathbf{p}d\tau$ is the magnetic moment of the element $d\tau$, then $-\dfrac{\mathbf{p} \cdot \mathbf{r}}{r^3}d\tau$ is the magnetic potential at the point A, so that $-\mathbf{f} \mid \nabla$ is the magnetic potential if $\mathbf{p}$ denotes the magnetic moment per unit volume. $-\dfrac{\mathbf{p} \times \mathbf{r}}{r^3}d\tau$ is the magnetic field strength which is set up by a steady elementary current $\mathbf{p}d\tau$ at the point A. Hence $-\mathbf{f} \mid \nabla$ is the magnetic field strength when $\mathbf{p}$ denotes the current strength per unit volume which is distributed over the region.

The distance r may equally be regarded as the position function of the position of A as the position function of the element $d\tau$. In the former case we found

$$\frac{1}{r}\Big|\nabla = \Big|\frac{\mathbf{r}}{r^3}.$$

In the second case let $\overline{\nabla}$ denote the corresponding operator. Then:

$$\frac{1}{r}\Big|\overline{\nabla} = -\Big|\frac{\mathbf{r}}{r^3}.$$

For since the terminal points of the vector $\mathbf{r}$ interchange rôles if $\overline{\nabla}$ is taken in place of ∇, it follows that $\mathbf{r}$ is to be changed into $-\mathbf{r}$. Accordingly in the integrals for

$$f \mid \nabla, \quad \mathbf{f} \mid \nabla, \quad \mathbf{F} \mid \nabla$$

we may replace $\frac{1}{r}\Big|\nabla$ by $-\frac{1}{r}\Big|\overline{\nabla}$ and we get

$$f \mid \nabla = -\int \rho\left(\frac{1}{r}\Big|\overline{\nabla}\right) d\tau,$$

$$\mathbf{f} \mid \nabla = -\int \mathbf{p}\left(\frac{1}{r}\Big|\overline{\nabla}\right) d\tau,$$

$$\mathbf{F} \mid \nabla = -\int \mathbf{P}\left(\frac{1}{r}\Big|\overline{\nabla}\right) d\tau.$$

Now

$$\rho\left(\frac{1}{r}\Big|\overline{\nabla}\right) = \frac{\rho}{r}\Big|\overline{\nabla} - \frac{1}{r}(\rho \mid \overline{\nabla}),$$

and analogously

$$\mathbf{p}\left(\frac{1}{r}\Big|\overline{\nabla}\right) = \frac{\mathbf{p}}{r}\Big|\overline{\nabla} - \frac{1}{r}(\mathbf{p} \mid \overline{\nabla}),$$

$$\mathbf{P}\left(\frac{1}{r}\Big|\overline{\nabla}\right) = \frac{\mathbf{P}}{r}\Big|\overline{\nabla} - \frac{1}{r}(\mathbf{P} \mid \overline{\nabla}).$$

It follows that each integral may be expressed as the difference of two integrals, for example:

$$f \mid \nabla = \int \frac{\rho \mid \overline{\nabla}}{r} d\tau - \int \frac{\rho}{r} \mid \overline{\nabla}\, d\tau.$$

In the other two cases instead of ρ, $\mathbf{p}$ or $\mathbf{P}$ are to be written. The integrals are to be taken throughout the whole region.

We will now assume, however, that if the integration is extended over a large enough region an approximation may be obtained to any degree of accuracy. The second integral, from the foregoing theorems, may be transformed into an integral over the boundary of the region and we would find, for example

$$f \mid \nabla = \int \frac{\rho \mid \overline{\nabla}}{r} d\tau + \int \frac{\rho}{r} d\mathfrak{G}.$$

We will assume further, that the functions ρ, **p**, **P** become so small at all points sufficiently far distant, that the surface integrals

$$\int \frac{\rho}{r}\, d\mathfrak{a}, \quad \int \frac{\mathbf{p}}{r}\, d\mathfrak{a}, \quad \int \frac{\mathbf{P}}{r}\, d\mathfrak{a}$$

become arbitrarily small for a sufficiently large region of integration. In effect this merely requires that ρ, **p**, **P** shall be of a higher order of smallness than $\frac{1}{r}$ at great distances.

On this assumption it follows that:

$$f \mid \nabla = \int \frac{\rho \mid \overline{\nabla}}{r}\, d\tau,$$

$$\mathbf{f} \mid \nabla = \int \frac{\mathbf{p} \mid \overline{\nabla}}{r}\, d\tau,$$

$$\mathbf{F} \mid \nabla = \int \frac{\mathbf{P} \mid \overline{\nabla}}{r}\, d\tau.$$

Each of these three integrals may therefore be expressed in dual form. The second form is the same as those of the integrals f, **f**, **F** themselves in which the first power of r occurs in the denominator, and a function of the position of the element $d\tau$ in the numerator independent of the position of A.

The magnetic field strength $-\mathbf{F} \mid \nabla$ of a current of density $\mid \mathbf{P} = \mathbf{p}$ distributed over the region is also capable of being thrown into the form

$$-\int \frac{\mathbf{P} \mid \overline{\nabla}}{r}\, d\tau \quad \text{or} \quad \int \frac{\overline{\nabla} \times \mathbf{p}}{r}\, d\tau,$$

and the magnetic potential in the form

$$-\int \frac{\mathbf{p} \mid \overline{\nabla}}{r}\, d\tau.$$

If **P** and **p** are complements of each other then the integrals **F** and **f** are also complements of each other. Hence instead of $\mathbf{f} \mid \nabla$ we may write $\mid \mathbf{F} \mid \nabla$, or what is the same, $\mathbf{F}\nabla$, or even $\nabla\mathbf{F}$, and instead of $\mathbf{F} \mid \nabla$ we may write the complement of $\mathbf{f}\nabla$, or what is the same, $-\nabla \times \mathbf{f}$.

Similarly under the sign of integration we may replace $\mathbf{p} \mid \overline{\nabla}$ by $\overline{\nabla}\mathbf{P}$, and $\mathbf{P} \mid \overline{\nabla}$ by $- \overline{\nabla} \times \mathbf{p}$.

Accordingly

$$\nabla \cdot \mathbf{f} = \int \frac{\overline{\nabla} \cdot \mathbf{p}}{r}\, d\tau,$$

or

$$\nabla \mathbf{F} = \int \frac{\overline{\nabla} \mathbf{P}}{r}\, d\tau,$$

and

$$\nabla \times \mathbf{f} = \int \frac{\overline{\nabla} \times \mathbf{p}}{r}\, d\tau.$$

CHAPTER III

TENSORS

§ 1. The Affine Transformation of Space

LET **a**, **b**, **c** be three mutually independent vectors so that an arbitrary vector **r** may be derived numerically from them:

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}.$$

Suppose the vector **r** drawn from a fixed point O to a variable point R so that the coefficients x, y, z of the vector simultaneously provide the co-ordinates of the point R referred to the unit vectors **a**, **b**, **c**.

Now imagine a space transformation such that the point O remains fixed, but the point R transforms into a new point R′. This will be reached from the point O by the vector

$$\mathbf{r}' = x\mathbf{e} + y\mathbf{f} + z\mathbf{g},$$

where x, y, z are the same coefficients as in the expression for **r** in terms of **a**, **b**, **c**, and **e**, **f**, **g** are three numerically independent vectors otherwise quite arbitrary.

This is described as an affine transformation of space. The vector joining a point R to any other point S is connected with the vector joining R′ to S′, the points into which R and S are transformed, in the same way as **r** is connected with **r**′.

For if the vectors OR and OS are denoted by

$$x_r\mathbf{a} + y_r\mathbf{b} + z_r\mathbf{c}$$

and

$$x_s\mathbf{a} + y_s\mathbf{b} + z_s\mathbf{c},$$

then RS = OS − OR is expressed by

$$(x_s - x_r)\mathbf{a} + (y_s - y_r)\mathbf{b} + (z_s - z_r)\mathbf{c},$$

and in analogous manner R′S′ is expressed by

$$(x_s - x_r)\mathbf{e} + (y_s - y_r)\mathbf{f} + (z_s - z_r)\mathbf{g},$$

i.e. the same coefficients that derive RS from **a**, **b**, **c** derive R′S′ from **e**, **f**, **g**, or as we may state it otherwise, in an affine transformation of space a vector derived numerically from **a**, **b**, **c** is transformed into a vector derived numerically from **e**, **f**, **g** by the same coefficients. Four points P, Q, R, S, determining a parallelogram, must after transformation again determine a parallelogram. For if the same vector joins P to Q as joins R to S, then the transformed vector must simultaneously join P′ to Q′ and R′ to S′. More generally we may say, if $\mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_n$ are any system of vectors among which a vector equation

$$a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + \ldots + a_n\mathbf{p}_n = 0$$

holds, then the same vector equation

$$a_1\mathbf{p}_1' + a_2\mathbf{p}_2' + \ldots + a_n\mathbf{p}_n' = 0$$

must hold between the transformed vectors $\mathbf{p}_1', \mathbf{p}_2' \ldots \mathbf{p}_n'$. For if $\mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_n$ are expressed in terms of **a**, **b**, **c** then the vector equation in **a**, **b**, **c** must be identically satisfied, since it has been assumed that **a**, **b**, **c** are numerically independent of each other. It is consequently also satisfied if **a**, **b**, **c** are replaced by **e**, **f**, **g**, that is to say, when we pass to the transformed vector equation.

If in particular two distances have the same direction but any arbitrary ratio of lengths the transformed distances will continue to have identical directions and bear the same ratio in lengths. For if $\mathbf{p}_1$ and $\mathbf{p}_2$ are the vectors in question with the same direction and if the ratio of their lengths is a, then

$$\mathbf{p}_1 = a\mathbf{p}_2$$

and the same equation must hold between the transformed vectors. Three points which initially lie in a straight line must remain in a straight line after transformation, and the ratio of the parts must remain unaltered.

Four points in a plane must after transformation remain in a plane. If from any one of the four points, three vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are drawn to the remaining points, then

$$\mathbf{p}_3 = a_1\mathbf{p}_1 + a_2\mathbf{p}_2,$$

and this relation holds also for the transformed points, i.e. if the first point is chosen as the origin of a co-ordinate system and $\mathbf{p}_1$ and $\mathbf{p}_2$ are made unit vectors leading to two of the other points, then a_1 and a_2 are the co-ordinates of the last point of the system (fig. 28). If the same construction is carried through for the transformed points, then the co-ordinates in the transformed system remain the same. The parallels which are drawn through the fourth point, to the lines joining the first point to the second and third, cut off on these lines lengths whose ratios to the connecting lines remain unaltered by the transformation.

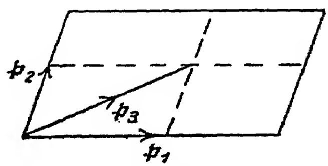

FIG. 28.

An analogous construction may be conducted in space. Consider five points. Let one be chosen as origin and draw vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$ to the remaining four points, where $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ are assumed initially to be independent of each other. Then

$$\mathbf{p}_4 = a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + a_3\mathbf{p}_3$$

and a_1, a_2, a_3 are the co-ordinates of the fifth point in a system whose origin lies in one point and whose unit vectors join the origin to the second, third, and fourth points. The same construction with the transformed points leads to the same co-ordinates a_1, a_2, a_3 of the transformed fifth point in the transformed system.

We may construct a picture of the transformation if we imagine a lattice work constructed from any three mutually independent vectors radiating from a fixed point O. In the transformation this lattice work is transformed into another in which each cell of the first corresponds to a cell of the second. The position of a point in a cell may be supposed

specified by planes passing through it parallel to the sides of the cell, and intersecting the edges of the cell. The ratio of the segments to the lengths of the edges determine the position of the point. These ratios, however, remain unaltered during the transformation. In particular we may consider the space-lattice of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ with which the definition of the affine transformation was originally associated. Its space-lattice transforms into that of the vectors $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$, and the vectorial equation

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

between the vectors $\mathbf{r}$, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ holds likewise between the transformed vectors

$$\mathbf{r}' = x\mathbf{e} + y\mathbf{f} + z\mathbf{g},$$

the coefficients x, y, z being unaffected. We term $\mathbf{r}'$ a vector function of the variable vector $\mathbf{r}$.

By $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ we denote as before the vectors reciprocal to the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, so that

$$\mathbf{a}^* = \frac{\mathbf{b} \times \mathbf{c}}{\mathbf{abc}}, \quad \mathbf{b}^* = \frac{\mathbf{c} \times \mathbf{a}}{\mathbf{abc}}, \quad \mathbf{c}^* = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{abc}},$$

and let us construct the expression

$$\mathbf{e}(\mathbf{a}^* . \mathbf{l}) + \mathbf{f}(\mathbf{b}^* . \mathbf{l}) + \mathbf{g}(\mathbf{c}^* . \mathbf{l}),$$

where $\mathbf{l}$ denotes any vector whatever. We may then say that this expression represents the affine transformation. For if in place of $\mathbf{l}$ the vector $\mathbf{a}$ is inserted, the terms $\mathbf{a} . \mathbf{b}^*$ and $\mathbf{a} . \mathbf{c}^*$ vanish, while $\mathbf{a} . \mathbf{a}^* = 1$. It therefore reduces to $\mathbf{e}$, and similarly $\mathbf{f}$ and $\mathbf{g}$ are derived when $\mathbf{b}$ and $\mathbf{c}$ are inserted. If now the vector

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$$

is introduced in place of $\mathbf{l}$, the sum

$$\mathbf{e}x + \mathbf{f}y + \mathbf{g}z$$

is derived, that is to say the vector $\mathbf{r}'$ results.

The expression

$$T = \mathbf{e}(\mathbf{a}^* . \mathbf{l}) + \mathbf{f}(\mathbf{b}^* . \mathbf{l}) + \mathbf{g}(\mathbf{c}^* . \mathbf{l}),$$

for reasons to be explained later, we term a "Tensor" and denote it by the letter **T**.

For simplification in writing, the complete form of the expression will be dropped and we will merely write

$$\mathbf{T} = \mathbf{ea}^* + \mathbf{fb}^* + \mathbf{gc}^*,$$

where, however, it must be remarked that the second factors $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ stand in place of $\mathbf{a}^* . \mathbf{1}$, $\mathbf{b}^* . \mathbf{1}$, $\mathbf{c}^* . \mathbf{1}$, that is by inserting $\mathbf{r}$ they lead to the coefficients, $\mathbf{a}^* . \mathbf{r}$, $\mathbf{b}^* . \mathbf{r}$, $\mathbf{c}^* . \mathbf{r}$. For any particular form of requirement that **T** may have to satisfy, it is always possible if necessary to return to the complete expression. It is evident, that the order of the three terms in **T** may be arbitrarily altered without changing in any way the full expression which we call $\mathbf{T} . \mathbf{r}$. Every term, moreover, may be broken up into the sum of an arbitrary number of other terms. For example,

if
$$\mathbf{a}^* = \mathbf{p} + \mathbf{q}$$
$$\begin{aligned}\mathbf{ea}^* &= \mathbf{e}(\mathbf{p} + \mathbf{q}) = \mathbf{e}[\mathbf{1} . (\mathbf{p} + \mathbf{q})] \\ &= \mathbf{e}(\mathbf{1} . \mathbf{p}) + \mathbf{e}(\mathbf{1} . \mathbf{q}) \\ &= \mathbf{ep} + \mathbf{eq},\end{aligned}$$

or if
$$\mathbf{e} = \mathbf{p}' + \mathbf{q}'$$
$$\begin{aligned}\mathbf{ea}^* &= (\mathbf{p}' + \mathbf{q}')(\mathbf{1} . \mathbf{a}^*) = \mathbf{p}'(\mathbf{1} . \mathbf{a}^*) + \mathbf{q}'(\mathbf{1} . \mathbf{a}^*) \\ &= \mathbf{p}'\mathbf{a}^* + \mathbf{q}'\mathbf{a}^*,\end{aligned}$$

or else simultaneously

$$\mathbf{ea}^* = (\mathbf{p}' + \mathbf{q}')(\mathbf{p} + \mathbf{q}) = \mathbf{p}'\mathbf{p} + \mathbf{q}'\mathbf{p} + \mathbf{p}'\mathbf{q} + \mathbf{q}'\mathbf{q}.$$

If now the second factor in each of the terms denote expressions $\mathbf{1} . \mathbf{p}$, $\mathbf{1} . \mathbf{q}$, then $\mathbf{T} . \mathbf{r}$ retains its value $\mathbf{r}'$. If $\mathbf{a}^* = m\mathbf{p}$, the numerical factor m may also be associated with **e**

$$\mathbf{ea}^* = \mathbf{e}(m\mathbf{p}) = (m\mathbf{e})\mathbf{p},$$

without $\mathbf{T} . \mathbf{r}$ undergoing any change.

In this manner, instead of $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$, three other mutually independent vectors **o**, **p**, **q** may be introduced by inserting for $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ the expression obtained by deriving them numerically from **o**, **p**, **q**. If the nine terms that arise are then arranged in the sequence **o**, **p**, **q** we have three terms

associated with $\mathbf{o}$ as a second factor which may again be combined into the one term

$$\mathbf{uo}$$

when $\mathbf{o}$ is taken outside the bracket. Similar expressions are derived for $\mathbf{p}$ and $\mathbf{q}$ so that $\mathbf{T}$ assumes the form

$$\mathbf{T} = \mathbf{uo} + \mathbf{vp} + \mathbf{wq}.$$

Here $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are the vectors which arise when the vectors reciprocal to $\mathbf{o}$, $\mathbf{p}$, $\mathbf{q}$ are changed by the transformation $\mathbf{T}$. It appears then that the terms $\mathbf{ea}^*$, $\mathbf{fb}^*$, $\mathbf{gc}^*$ may be dealt with as far as the distributive law is concerned in the expression for $\mathbf{T}$ as if they were products, except that the factors must not be interchanged, for unlike the case of scalar or of vectorial products of two vectors, two totally different results would be obtained. The same transformation $\mathbf{T}$ may be represented as the sum of three or more than three terms $\mathbf{uo}$ in a number of ways. As soon, however, as three definite independent vectors are introduced for the second factor of each term, from which vectors the second factors are numerically derivable, then the first factors also transform into three definite vectors, namely, into the three vectors which arise when the reciprocal vectors of the original are transformed. Let there be two different expressions for $\mathbf{T}$:

$$\mathbf{T} = \mathbf{uo} + \mathbf{vp} + \mathbf{wq}$$

and

$$\mathbf{T} = \mathbf{u'p'} + \mathbf{v'q'} + \ldots$$

where the second expression may have more than three terms, and let a vectorial area be derived from both expressions by regarding the products $\mathbf{uo}$, $\mathbf{u'p'}$, etc., as the external products of vectors, then in both cases the same vectorial area must originate.

For if in the expression

$$\mathbf{u'p'} + \mathbf{v'q'} + \ldots$$

$\mathbf{p'}$, $\mathbf{q'}$ are expressed in terms of $\mathbf{o}$, $\mathbf{p}$, $\mathbf{q}$ and the brackets removed according to the distributive law, then, as we have

seen above, on arranging according to the sequence **o, p, q**, the expression:

$$\mathbf{uo} + \mathbf{vp} + \mathbf{wq}$$

must arise, where **u, v, w** are uniquely determined as the vectors to which **o*, p*, q***, i.e. to which the vectors reciprocal to **o, p, q**, are changed by the transformation. But all these operations are equally valid when we deal with the external products of two vectors. Hence the vectorial area

$$\mathbf{u}'\mathbf{p}' + \mathbf{v}'\mathbf{q}' + \ldots$$

must be equal to the vectorial area

$$\mathbf{uo} + \mathbf{vp} + \mathbf{wq}.$$

If, on the other hand, two scalar quantities are derived from the two expressions by replacing the products **uo, u'p'**, etc., by the scalar products **u . o, u' . p'**, etc., these two scalars are also equal. For the operations by means of which the two expressions are transformed into each other are valid also for scalar products, and consequently the value

$$\mathbf{u}' \cdot \mathbf{p}' + \mathbf{v}' \cdot \mathbf{q}' + \ldots$$

must be equal to the value

$$\mathbf{u} \cdot \mathbf{o} + \mathbf{v} \cdot \mathbf{p} + \mathbf{w} \cdot \mathbf{q}.$$

This value, as with the vectorial area

$$\mathbf{uo} + \mathbf{vp} + \mathbf{wq},$$

must have a geometrical interpretation for the transformation, which will explain why both are independent of the form of expression of the transformation. A closer investigation of the transformation will reveal the interpretation.

§ 2. Conjugate Tensors

If the factors are interchanged in every term, in general the transformation will be changed. For example

$$\mathbf{ou} + \mathbf{pv} + \mathbf{qw}$$

will not represent the same transformation as

$$\mathbf{T} = \mathbf{uo} + \mathbf{vp} + \mathbf{wq}.$$

The one changes the vectors reciprocal to **o**, **p**, **q** into **u**, **v**, **w**, while the other changes those reciprocal to **u**, **v**, **w** into **o**, **p**, **q**.

These two transformations, and likewise the two tensors, are referred to as being "conjugate" to one another, and we write

$$\overline{\mathbf{T}} = \mathbf{ou} + \mathbf{pv} + \mathbf{qw},$$

if we put

$$\mathbf{T} = \mathbf{uo} + \mathbf{vp} + \mathbf{wq}.$$

In general $\mathbf{T}$ and $\overline{\mathbf{T}}$ are different from each other. The special case may arise where the two conjugate transformations become identical. Such tensors constitute a very special class which must receive special investigation, and, as we shall see later, has its own particular applications. They are termed *symmetrical* tensors.

If a tensor

$$\mathbf{T} = \mathbf{uo} + \mathbf{vp} + \mathbf{wq}$$

with a vector **r**, leads to a new vector

$$\mathbf{u}(\mathbf{o}\,.\,\mathbf{r}) + \mathbf{v}(\mathbf{p}\,.\,\mathbf{r}) + \mathbf{w}(\mathbf{q}\,.\,\mathbf{r}),$$

then this is denoted by $\mathbf{T}\,.\,\mathbf{r}$ or $\mathbf{T} \mid \mathbf{r}$.

Since $\mid \mathbf{r}$ denotes a vectorial area **R**, we may also write at once:

$$\mathbf{T}\,.\,\mathbf{r} = \mathbf{TR} = \mathbf{u}(\mathbf{oR}) + \mathbf{v}(\mathbf{pR}) + \mathbf{w}(\mathbf{qR}).$$

In other words, it is immaterial whether we construct the scalar products with the vector **r** or the external products with the representation **R**. As will become clear in later applications it is well to have recourse to both possibilities.

The scalar product of the transformed vector $\mathbf{T}\,.\,\mathbf{r}$ with the vector **r** is equal to:

$$(\mathbf{T}\,.\,\mathbf{r})\,.\,\mathbf{r} = (\mathbf{u}\,.\,\mathbf{r})(\mathbf{o}\,.\,\mathbf{r}) + (\mathbf{v}\,.\,\mathbf{r})(\mathbf{p}\,.\,\mathbf{r}) + (\mathbf{w}\,.\,\mathbf{r})(\mathbf{q}\,.\,\mathbf{r}).$$

The same scalar function is derived from the scalar product $(\overline{\mathbf{T}} \, . \, \mathbf{r}) \, . \, \mathbf{r}$:

$$(\overline{\mathbf{T}} \, . \, \mathbf{r}) \, . \, \mathbf{r} = (\mathbf{o} \, . \, \mathbf{r})(\mathbf{u} \, . \, \mathbf{r}) + (\mathbf{p} \, . \, \mathbf{r})(\mathbf{v} \, . \, \mathbf{r}) + (\mathbf{q} \, . \, \mathbf{r})(\mathbf{w} \, . \, \mathbf{r}).$$

Since no confusion need be feared we will omit the bracket in the expression:

$$(\mathbf{T} \, . \, \mathbf{r}) \, . \, \mathbf{r}.$$

Let us now construct the gradient of this scalar function, noting that the gradient of the scalar product of a constant vector with a variable position vector $\mathbf{r}$ is equal to the constant vector. For the gradient of a scalar function f is defined by the equation

$$df = \nabla f \, . \, d\mathbf{r},$$

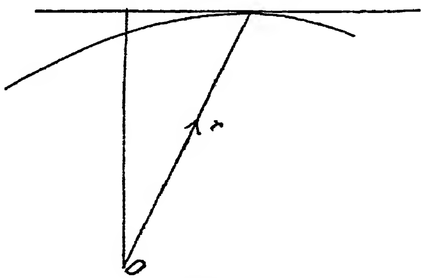

FIG. 29.

and also

$$d(\mathbf{c} \, . \, \mathbf{r}) = \mathbf{c} \, . \, d\mathbf{r},$$

so that

$$\nabla(\mathbf{c} \, . \, \mathbf{r}) = \mathbf{c}$$

and

$$\nabla(\mathbf{u} \, . \, \mathbf{r})(\mathbf{o} \, . \, \mathbf{r}) = \mathbf{u}(\mathbf{o} \, . \, \mathbf{r}) + (\mathbf{u} \, . \, \mathbf{r})\mathbf{o}.$$

Consequently the gradient of $\mathbf{T} \, . \, \mathbf{r} \, . \, \mathbf{r}$, or what is the same, the gradient of $\overline{\mathbf{T}} \, . \, \mathbf{r} \, . \, \mathbf{r}$ is equal to

$$\mathbf{u}(\mathbf{o} \, . \, \mathbf{r}) + (\mathbf{u} \, . \, \mathbf{r})\mathbf{o} + \mathbf{v}(\mathbf{p} \, . \, \mathbf{r}) + (\mathbf{v} \, . \, \mathbf{r})\mathbf{p} + \mathbf{w}(\mathbf{q} \, . \, \mathbf{r}) + (\mathbf{w} \, . \, \mathbf{r})\,\mathbf{q},$$

i.e.

$$\nabla(\mathbf{T} \, . \, \mathbf{r} \, . \, \mathbf{r}) = \nabla(\overline{\mathbf{T}} \, . \, \mathbf{r} \, . \, \mathbf{r}) = \mathbf{T} \, . \, \mathbf{r} + \overline{\mathbf{T}} \, . \, \mathbf{r}.$$

If the scalar function $\mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r}$ is set equal to a constant, we have the equation to a surface of the second degree. For this purpose we must suppose $\mathbf{r}$ set off from the origin O in each direction leading to a point on the surface and of the length demanded by the constant. The gradient is then at right angles to the surface at the point considered. The scalar product of the gradient with $\mathbf{r}$ gives

$$\nabla(\mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r})\,.\,\mathbf{r} = \mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r} + \overline{\mathbf{T}}\,.\,\mathbf{r}\,.\,\mathbf{r} = 2\mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r},$$

i.e. at all points on the surface the scalar product of the gradient with $\mathbf{r}$ has the same value. Now the scalar product equals the length of the gradient multiplied by the projection of $\mathbf{r}$ upon it. It follows that the length of the gradient must be inversely proportional to the distance of the point O from the tangent plane (fig. 29).

§ 3. Vectors which Transform into Themselves

To examine the properties of the transformation $\mathbf{T}$ it is proposed to consider whether there exist vectors $\mathbf{r}$ which on transformation merely reappear multiplied by a number λ, so that:

$$\mathbf{r}' = \lambda\mathbf{r}.$$

The vectors $\mathbf{o}^*$, $\mathbf{p}^*$, $\mathbf{q}^*$, reciprocal to $\mathbf{o}$, $\mathbf{p}$, $\mathbf{q}$, become on transformation $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. Hence if $\mathbf{r}$ is derived numerically from $\mathbf{o}^*$, $\mathbf{p}^*$, $\mathbf{q}^*$

$$\mathbf{r} = \mathbf{o}^*(\mathbf{r}\,.\,\mathbf{o}) + \mathbf{p}^*(\mathbf{r}\,.\,\mathbf{p}) + \mathbf{q}^*(\mathbf{r}\,.\,\mathbf{q}),$$

we get

$$\mathbf{r}' = \mathbf{T}\,.\,\mathbf{r} = \mathbf{u}(\mathbf{r}\,.\,\mathbf{o}) + \mathbf{v}(\mathbf{r}\,.\,\mathbf{p}) + \mathbf{w}(\mathbf{r}\,.\,\mathbf{q}).$$

The question is whether there are any vectors $\mathbf{r}$ for which

$$\begin{aligned}\lambda[\mathbf{o}^*(\mathbf{r}\,.\,\mathbf{o}) + \mathbf{p}^*(\mathbf{r}\,.\,\mathbf{p}) + \mathbf{q}^*(\mathbf{r}\,.\,\mathbf{q})]\\ = \mathbf{u}(\mathbf{r}\,.\,\mathbf{o}) + \mathbf{v}(\mathbf{r}\,.\,\mathbf{p}) + \mathbf{w}(\mathbf{r}\,.\,\mathbf{q}),\end{aligned}$$

or what amounts to the same thing,

$$(\lambda\mathbf{o}^* - \mathbf{u})(\mathbf{r}\,.\,\mathbf{o}) + (\lambda\mathbf{p}^* - \mathbf{v})(\mathbf{r}\,.\,\mathbf{p}) + (\lambda\mathbf{q}^* - \mathbf{w})(\mathbf{r}\,.\,\mathbf{q}) = 0.$$

Since, on account of the assumed independence of **o**, **p**, **q**, the three coefficients **r** . **o**, **r** . **p**, and **r** . **q** cannot all vanish, this equation establishes a relation between the three vectors :

$$\lambda \mathbf{o}^* - \mathbf{u},\ \lambda \mathbf{p}^* - \mathbf{v},\ \lambda \mathbf{q}^* - \mathbf{w}.$$

From such a relation, however, it follows that the external product of these three vectors must be zero. Expanding the external product in powers of λ, an equation of the third degree in λ is derived :

$$\mathbf{o}^*\mathbf{p}^*\mathbf{q}^*\lambda^3 - (\mathbf{p}^*\mathbf{q}^*\mathbf{u} + \mathbf{q}^*\mathbf{o}^*\mathbf{v} + \mathbf{o}^*\mathbf{p}^*\mathbf{w})\lambda^2$$
$$+ (\mathbf{o}^*\mathbf{v}\mathbf{w} + \mathbf{p}^*\mathbf{w}\mathbf{u} + \mathbf{q}^*\mathbf{u}\mathbf{v})\lambda - \mathbf{u}\mathbf{v}\mathbf{w} = 0,$$

or dividing by $\mathbf{o}^*\mathbf{p}^*\mathbf{q}^*$, returning to the vectors **o**, **p**, **q** in the second term and introducing the vectors $\mathbf{u}^*$, $\mathbf{v}^*$, $\mathbf{w}^*$ reciprocal to **u**, **v**, **w** in the third term,

$$\lambda^3 - (\mathbf{o} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{v} + \mathbf{q}\mathbf{w})\lambda^2$$
$$+ \frac{\mathbf{u}\mathbf{v}\mathbf{w}}{\mathbf{o}^*\mathbf{p}^*\mathbf{q}^*}[(\mathbf{o}^* \cdot \mathbf{u}^* + \mathbf{p}^* \cdot \mathbf{v}^* + \mathbf{q}^* \cdot \mathbf{w}^*)\lambda - 1] = 0.$$

We note that the coefficients of this equation are composed of the three expressions :

$$\mathbf{o} \cdot \mathbf{u} + \mathbf{p} \cdot \mathbf{v} + \mathbf{q} \cdot \mathbf{w},$$
$$\mathbf{o}^* \cdot \mathbf{u}^* + \mathbf{p}^* \cdot \mathbf{v}^* + \mathbf{q} \quad \cdot \mathbf{w}^*,$$
$$\frac{\mathbf{u}\mathbf{v}\mathbf{w}}{\mathbf{o}^*\mathbf{p}^*\mathbf{q}^*}.$$

Of these three expressions the first has already been met with, for it arose from the transformation

$$T = \mathbf{u}\mathbf{o} + \mathbf{v}\mathbf{p} + \mathbf{w}\mathbf{q}$$

when the three terms were replaced by the scalar products. The second expression originated in a like manner from the transformation :

$$T^* = \mathbf{o}^*\mathbf{u}^* + \mathbf{p}^*\mathbf{v}^* + \mathbf{q}^*\mathbf{w}^*.$$

This transformation changes the vectors **u**, **v**, **w** into

$\mathbf{o}^*$, $\mathbf{p}^*$, $\mathbf{q}^*$, that is, reverses the transformation **T**. The third expression is the ratio of the volumes of the two parallelepiped **uvw** and $\mathbf{o}^*\mathbf{p}^*\mathbf{q}^*$, which are changed into each other by the transformations **T** and $\mathbf{T}^*$.

$$\frac{\mathbf{uvw}}{\mathbf{o}^*\mathbf{p}^*\mathbf{q}^*} = (\mathbf{uvw})(\mathbf{opq}).$$

Moreover, this expression must be independent of the choice of the three vectors, **o**, **p**, **q**. The quotient which has any arbitrary volume as numerator, and the same region transformed as denominator, always retains the same value.

If λ is a real root of the equation and is inserted in the expressions

$$\lambda\mathbf{o}^* - \mathbf{u},\ \lambda\mathbf{p}^* - \mathbf{v},\ \lambda\mathbf{q}^* - \mathbf{w},$$

then since the external product of these three vectors vanishes, they are mutually dependent, and since the relation between them is represented by a vectorial equation, the ratios of $\mathbf{r}\,.\,\mathbf{o}$, $\mathbf{r}\,.\,\mathbf{p}$, $\mathbf{r}\,.\,\mathbf{q}$ are given by the coefficients, and they provide, to an arbitrary factor of proportionality, the coefficients of the required vector **r** referred to the vectors $\mathbf{o}^*$, $\mathbf{p}^*$, $\mathbf{q}^*$. The proportionality factor must remain arbitrary, for if the vector **r** has the desired property that it transforms into $\lambda\mathbf{r}$, then an arbitrary positive or negative multiple of **r** possesses the same property.

As we have already remarked, the three vectors **o**, **p**, **q** may be arbitrarily selected, provided they are mutually independent. We will assume that they constitute a right-handed orthogonal system and are each of unit length. The reciprocal vectors $\mathbf{o}^*$, $\mathbf{p}^*$, $\mathbf{q}^*$ are then identical with **o**, **p**, **q** and $\mathbf{opq} = 1$.

Three vectors possessing these characteristics will be denoted by the letters **i**, **j**, **k**. The equation of the third degree in λ may then be written directly in the form:

$$\lambda^3 - (\mathbf{u}\,.\,\mathbf{i} + \mathbf{v}\,.\,\mathbf{j} + \mathbf{w}\,.\,\mathbf{k})\lambda^2 + \mathbf{uvw}[(\mathbf{u}^*\,.\,\mathbf{i} + \mathbf{v}^*\,.\,\mathbf{j} + \mathbf{w}^*\,.\,\mathbf{k})\lambda - 1] = 0.$$

§ 4. Rotation Tensors

If the transformation consists of a rotation about an axis passing through O, then **u**, **v**, **w** must also be a right-handed orthogonal system, each member being of unit length. The reciprocal vectors **u***, **v***, **w*** are also then identical with **u**, **v**, **w** and **uvw** = 1.

Consequently we will denote them by **i**′, **j**′, **k**′. The equation of the third degree in λ then assumes the form :

$$\lambda^3 - (\mathbf{i}' . \mathbf{i} + \mathbf{j}' . \mathbf{j} + \mathbf{k}' . \mathbf{k})(\lambda^2 - \lambda) - 1 = 0.$$

One root is evidently $\lambda = 1$. The corresponding vector **r** will therefore remain unaltered by the transformation, i.e. it must lie along the axis of rotation. The ratios of the coefficients (**r** . **i**), (**r** . **j**), (**r** . **k**) of such a vector we derive from the vector equation :

$$(\mathbf{i} - \mathbf{i}')(\mathbf{r} . \mathbf{i}) + (\mathbf{j} - \mathbf{j}')(\mathbf{r} . \mathbf{j}) + (\mathbf{k} - \mathbf{k}')(\mathbf{r} . \mathbf{k}) = 0.$$

In point of fact, **i** − **i**′ may be represented by the line joining the end point of **i** to that of **i**′, if both are drawn from the point O. The vector **i** − **i**′ is thus perpendicular to the axis of rotation, and a similar argument applies to **j** − **j**′ and **k** − **k**′. All three are therefore parallel to the same plane and consequently mutually dependent.

We can simplify the solution by assuming **i** to lie in the axis of rotation. Then **i**′ = **i**, and **j**, **k**, **j**′, **k**′ are parallel to the same plane.

Moreover

$$\mathbf{i} . \mathbf{i}' = 1, \quad \mathbf{j} . \mathbf{j}' = \mathbf{k} . \mathbf{k}' = \cos\theta,$$

where θ represents the angle of rotation (fig. 30). Hence we obtain

$$\mathbf{i}' . \mathbf{i} + \mathbf{j}' . \mathbf{j} + \mathbf{k}' . \mathbf{k} = 1 + 2\cos\theta.$$

Since the sum of the scalar products, as we saw above, always has the same value irrespective of the form in which the transformation may be given, this provides a simple method of calculating the angle of rotation θ. It is merely necessary to regard all the terms in the expression

for the transformation as scalar products of the two factors, and the sum is the value of $1 + 2 \cos \theta$.

The axis of rotation may be determined in a similar manner. We have just seen that the vectorial area, derived by regarding all the terms in the expression for the transformation as external products of the two factors, always remains the same irrespective of the form adopted by the expression for the transformation. The geometrical interpretation of the vectorial area may consequently be found by assuming $\mathbf{i}$ again to lie in the axis of rotation. Then

$$\mathbf{i}'\mathbf{i} = 0 \text{ and } \mathbf{j}'\mathbf{j} = \mathbf{k}'\mathbf{k},$$

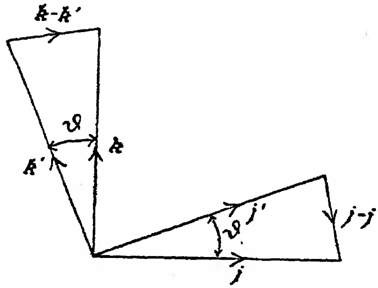

FIG. 30.

the latter being a vectorial area perpendicular to the axis of rotation of numerical value equal to sin θ and whose *sense* for $\theta < 180°$ is opposite to the direction of rotation. (Cf. fig. 30.) If we interchange the order of the factors, the *sense* coincides with the direction of rotation. The vectorial area

$$\mathbf{ou} + \mathbf{pv} + \mathbf{qw}$$

represents the rotational transformation irrespective of the choice of $\mathbf{o}$, $\mathbf{p}$, $\mathbf{q}$, in so far as this vectorial area stands perpendicularly to the axis of rotation; its *sense* is the sense of the rotation and its numerical value is equal to $2 \sin \theta$, where θ is the angle of rotation taken smaller than 180°.

There is indicated, then, the existence of a special rotation transformation such that every self-reciprocal system of

three vectors is transformed into another self-reciprocal system of the same sense. The conjugate transformation derived according to our previous definition by interchanging the factors in each term is again a rotational transformation. It changes the second system back again into the first. If the conjugate transformation is applied to $\mathbf{r}'$, we must derive $\mathbf{r}$ from it again. This is not always the case with other transformations. For if a transformation changes a self-reciprocal system $\mathbf{i}, \mathbf{j}, \mathbf{k}$ into a system $\mathbf{u}, \mathbf{v}, \mathbf{w}$, then the conjugate transformation changes $\mathbf{u}^*, \mathbf{v}^*, \mathbf{w}^*$—the system reciprocal to $\mathbf{u}, \mathbf{v}, \mathbf{w}$—into $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Accordingly, the conjugate transformation can only reverse the previous one if $\mathbf{u}^*, \mathbf{v}^*, \mathbf{w}^*$ is identical with $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

If this is the case, and if in addition $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a system with the same "sense" as $\mathbf{i}, \mathbf{j}, \mathbf{k}$, then the transformation is a rotation. If, however, $\mathbf{u}^*, \mathbf{v}^*, \mathbf{w}^*$ is identical with $\mathbf{u}, \mathbf{v}, \mathbf{w}$, but the "sense" of the system $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is opposite to that of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, then the case is not that of a simple rotational transformation. It can be replaced, however, by a rotational transformation combined with an "inversion." By "inversion" is meant a transformation which changes $\mathbf{r}$ into $-\mathbf{r}$, that is, it corresponds to rotation in the opposite direction. If we have two transformations which reverse each other, which are in fact "reciprocal to each other," we may say that the rotational transformations and those combined with inversion have, of all transformations, the special property that the conjugate transformation is also always the reciprocal transformation. (The inversion itself is to be included.)

§ 5. Self-Conjugate or Symmetrical Tensors

Associated with these consider yet another class of transformation characterised by the requirement that each tensor $\mathbf{T}$ is identical with its conjugate $\overline{\mathbf{T}}$.

As we found above, the gradient of the scalar function $\mathbf{T} \cdot \mathbf{r} \cdot \mathbf{r}$ or what is the same $\overline{\mathbf{T}} \cdot \mathbf{r} \cdot \mathbf{r}$ equals:

$$\mathbf{T} \cdot \mathbf{r} + \overline{\mathbf{T}} \cdot \mathbf{r}.$$

If then

$$\mathbf{T}\,.\,\mathbf{r} = \bar{\mathbf{T}}\,.\,\mathbf{r},$$

the transformation of $\mathbf{r}$ equals half the gradient of the scalar function $\mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r}$,

$$\mathbf{T}\,.\,\mathbf{r} = \tfrac{1}{2}\nabla(\mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r}).$$

The vectors $\mathbf{r}$, from which $\mathbf{T}\,.\,\mathbf{r}$ differs only by a constant factor λ, must therefore lie in the direction of the principal axis of the surface of the second order,

$$\mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r} = \text{constant},$$

where $\mathbf{r}$ is presumed drawn from O in order to reach the points on the surface. For the principal axes give the points on the surface where the gradient coincides with the direction of the radius vector. The roots of the equation of the third degree in λ must then have three real roots λ_1, λ_2, λ_3, and to these there correspond three mutually perpendicular vectors of arbitrary length which on transformation are multiplied by the factors λ_1, λ_2, λ_3. If a self-reciprocal system of vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is set up with these directions, then

$$\mathbf{u} = \lambda_1\mathbf{i},\ \mathbf{v} = \lambda_2\mathbf{j},\ \mathbf{w} = \lambda_3\mathbf{k},$$

and consequently

$$\mathbf{T} = \lambda_1\mathbf{ii} + \lambda_2\mathbf{jj} + \lambda_3\mathbf{kk}.$$

The cells of the cubical lattice work formed from $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are transformed into rectangular parallelepipeds whose edges are parallel to those of the cubical cells. If λ_1, λ_2, λ_3 are all three positive, the transformation of the space consists of a contraction ($\lambda < 1$) or expansion ($\lambda > 1$) of that space in the three mutually perpendicular directions. With negative values of λ there is in addition to the contraction or expansion in the corresponding direction a mirroring in a plane at right angles and passing through O.

We have presumed that the properties of the principal axes of a surface of the second degree are known. These may be investigated in terms of the notation developed for vector analysis in the following manner:

Let us imagine the various vectors $\mathbf{r}$ of length unity set off from the point O so that their end points lie on the surface of a sphere. Suppose each surface point covered by the value of the scalar function $f = \mathbf{T}.\mathbf{r}.\mathbf{r}$ where $\mathbf{r}$ is the vector to that point. The scalar function is finite and continuous over the surface of the sphere and must therefore attain its highest and its lowest values in at least one point in each case.

In passing from one such point to a neighbouring one on the surface of the sphere, the scalar function changes by

$$df = \nabla f \,.\, d\mathbf{r},$$

or, if dr, denotes the numerical value of $d\mathbf{r}$,

$$\frac{df}{dr} = \nabla f \,.\, \frac{d\mathbf{r}}{dr}.$$

$\frac{d\mathbf{r}}{dr}$ is a vector of unit length, which is at right angles to $\mathbf{r}$ at the point considered. Now $\frac{df}{dr}$ must be zero, for if it were otherwise it would have to change sign with $\frac{d\mathbf{r}}{dr}$, i.e. f would increase or diminish in moving in opposite directions on the spherical surface, and therefore it could possess neither a maximum nor a minimum value at the point in question. At those points where the upper or lower limit is reached, it follows therefore that the gradient ∇f is at right angles to the vector $\frac{d\mathbf{r}}{dr}$, that is to say, it must differ from $\mathbf{r}$ by a mere scalar factor. At both ends of any diameter of the sphere the scalar function $\mathbf{T}.\mathbf{r}.\mathbf{r}$ has the same value, for if $\mathbf{r}$ is changed into $-\mathbf{r}$ then $\mathbf{T}.\mathbf{r}$ changes into $-\mathbf{T}.\mathbf{r}$. Hence the upper and lower limits correspond at least to two different diameters of the sphere, for which

$$\mathbf{T}.\mathbf{r} = \lambda\mathbf{r}.$$

By scalar multiplication with $\mathbf{r}$, we get:

$$\mathbf{T}.\mathbf{r}.\mathbf{r} = \lambda\mathbf{r}.\mathbf{r} = \lambda.$$

Hence for the one diameter λ equals the upper limit of the scalar function and for the other diameter the lower limit. These two real and distinct values of λ—if for the present the case where $\mathbf{T}\,.\,\mathbf{r}\,.\,\mathbf{r}$ is constant over the whole surface of the sphere is omitted—must be roots of the previously derived equation of the third degree. Let them be denoted by λ_1 and λ_2, and the corresponding radii by $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{T}\,.\,\mathbf{r}_1 = \lambda_1\mathbf{r}_1\,;\ \mathbf{T}\,.\,\mathbf{r}_2 = \lambda_2\mathbf{r}_2.$$

If the first vector equation be multiplied scalarly by $\mathbf{r}_2$, and the second by $\mathbf{r}_1$, then:

$$\mathbf{T}\,.\,\mathbf{r}_1\,.\,\mathbf{r}_2 = \lambda_1\mathbf{r}_1\,.\,\mathbf{r}_2\,;\ \mathbf{T}\,.\,\mathbf{r}_2\,.\,\mathbf{r}_1 = \lambda_2\mathbf{r}_2\,.\,\mathbf{r}_1.$$

Now

$$\mathbf{T}\,.\,\mathbf{r}_1\,.\,\mathbf{r}_2 = (\mathbf{u}\,.\,\mathbf{r}_2)(\mathbf{o}\,.\,\mathbf{r}_1) + (\mathbf{v}\,.\,\mathbf{r}_2)(\mathbf{p}\,.\,\mathbf{r}_1) + (\mathbf{w}\,.\,\mathbf{r}_2)(\mathbf{q}\,.\,\mathbf{r}_1),$$

and since $\qquad \mathbf{T}\,.\,\mathbf{r}_1 = \overline{\mathbf{T}}\,.\,\mathbf{r}_1,$

$$\begin{aligned}\mathbf{T}\,.\,\mathbf{r}_1\,.\,\mathbf{r}_2 &= \overline{\mathbf{T}}\,.\,\mathbf{r}_1\,.\,\mathbf{r}_2\\ &= (\mathbf{o}\,.\,\mathbf{r}_2)(\mathbf{u}\,.\,\mathbf{r}_1) + (\mathbf{p}\,.\,\mathbf{r}_2)(\mathbf{v}\,.\,\mathbf{r}_1) + (\mathbf{q}\,.\,\mathbf{r}_2)(\mathbf{w}\,.\,\mathbf{r}_1).\end{aligned}$$

This is, however, equal to $\mathbf{T}\,.\,\mathbf{r}_2\,.\,\mathbf{r}_1$ and consequently

$$\mathbf{T}\,.\,\mathbf{r}_1\,.\,\mathbf{r}_2 = \mathbf{T}\,.\,\mathbf{r}_2\,.\,\mathbf{r}_1,$$

and therefore

$$\lambda_1\mathbf{r}_1\,.\,\mathbf{r}_2 = \lambda_2\mathbf{r}_1\,.\,\mathbf{r}_2.$$

Since λ_1 and λ_2 are different, however, this equation can exist only if:

$$\mathbf{r}_1\,.\,\mathbf{r}_2 = 0.$$

That is to say, the two diameters at whose ends the scalar function assumes its upper and lower limits are at right angles.

Now let

$$\mathbf{i} = \mathbf{r}_1,\ \mathbf{j} = \mathbf{r}_2,\ \mathbf{k} = \mathbf{r}_1 \times \mathbf{r}_2,$$

then $\qquad \mathbf{u} = \lambda_1\mathbf{i},\ \mathbf{v} = \lambda_2\mathbf{j}\,;$

hence $\qquad \mathbf{T} = \lambda_1\mathbf{ii} + \lambda_2\mathbf{jj} + \mathbf{wk},$

and $\qquad \overline{\mathbf{T}} = \lambda_1\mathbf{ii} + \lambda_2\mathbf{jj} + \mathbf{kw}.$

From the fact that $\mathbf{T} \cdot \mathbf{r} = \overline{\mathbf{T}} \cdot \mathbf{r}$ it follows that for every vector $\mathbf{r}$

$$\mathbf{w}(\mathbf{k} \cdot \mathbf{r}) = \mathbf{k}(\mathbf{w} \cdot \mathbf{r}),$$

i.e. the vectors $\mathbf{w}$ and $\mathbf{k}$ differ only by a scalar factor of proportionality.

If we write

$$\mathbf{w} = \lambda_3 \mathbf{k},$$

then
$$\mathbf{T} = \lambda_1 \mathbf{ii} + \lambda_2 \mathbf{jj} + \lambda_3 \mathbf{kk}.$$

The scalar function $f = \mathbf{T} \cdot \mathbf{r} \cdot \mathbf{r}$ accordingly assumes the form

$$f = \lambda_1(\mathbf{i} \cdot \mathbf{r})^2 + \lambda_2(\mathbf{j} \cdot \mathbf{r})^2 + \lambda_3(\mathbf{k} \cdot \mathbf{r})^2,$$

or if the projections of $\mathbf{r}$ on $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are denoted by x, y, z,

$$f = \lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2.$$

In the foregoing we have excluded the case where the upper and lower limits of the scalar function coincide. For the consideration of that case it is merely necessary to assume that f is constant over the whole surface of the sphere. Let this value of f be λ, then on the surface:

$$f = \mathbf{T} \cdot \mathbf{r} \cdot \mathbf{r} = \lambda \mathbf{r} \cdot \mathbf{r}.$$

But this same equation is valid over the whole space. For if $\mathbf{r}$ is transformed into $n\mathbf{r}$, then $\mathbf{T} \cdot \mathbf{r}$ becomes $n\mathbf{T} \cdot \mathbf{r}$. Since, however, $\mathbf{T} \cdot \mathbf{r}$ is equal to half the gradient of $\mathbf{T} \cdot \mathbf{r} \cdot \mathbf{r}$, and $\mathbf{r}$ is equal to half the gradient of $\mathbf{r} \cdot \mathbf{r}$, for the whole region

$$\mathbf{T} \cdot \mathbf{r} = \lambda \mathbf{r}.$$

For positive values of λ this represents a simple transformation preserving similarity, for negative values there is an inversion in addition. For $\lambda = -1$ it is a pure inversion. All these transformations may be expressed in the form

$$\mathbf{T} = \lambda(\mathbf{aa}^* + \mathbf{bb}^* + \mathbf{cc}^*),$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three arbitrary but mutually independent

vectors and $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ are their reciprocals. For if $\mathbf{r}$ be derived numerically from $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$,

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c},$$

then

$$(\mathbf{r} \cdot \mathbf{a}^*) = x; \ (\mathbf{r} \cdot \mathbf{b}^*) = y; \ (\mathbf{r} \cdot \mathbf{c}^*) = z;$$

thus

$$\mathbf{a}(\mathbf{r} \cdot \mathbf{a}^*) + \mathbf{b}(\mathbf{r} \cdot \mathbf{b}^*) + \mathbf{c}(\mathbf{r} \cdot \mathbf{c}^*) = \mathbf{r}$$

or

$$\mathbf{T} \cdot \mathbf{r} = \lambda\mathbf{r}.$$

If for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ we choose a self-reciprocal system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the tensor may be written in the form:

$$\mathbf{T} = \lambda(\mathbf{ii} + \mathbf{jj} + \mathbf{kk}).$$

If the upper and lower limits λ_1 and λ_2 are distinct, the special case may arise where λ_3 becomes equal to λ_1 or λ_2. If, for example, $\lambda_3 = \lambda_2$ then the vectors $\mathbf{j}$ and $\mathbf{k}$ transform into $\lambda_2\mathbf{j}$ and $\lambda_2\mathbf{k}$, so that each vector derived numerically from $\mathbf{j}$ and $\mathbf{k}$ is by the transformation multiplied by λ_2, and in place of $\mathbf{j}$ and $\mathbf{k}$ we may take $\mathbf{j}'$ and $\mathbf{k}'$ any two reciprocal vectors derived numerically from $\mathbf{j}$ and $\mathbf{k}$, together with $\mathbf{i}$, in which case we would likewise have:

$$\mathbf{T} = \lambda_1\mathbf{ii} + \lambda_2\mathbf{j}'\mathbf{j}' + \lambda_2\mathbf{k}'\mathbf{k}'.$$

If λ_1, λ_2, λ_3, on the other hand, are all distinct, there are only three diameters of the sphere for which:

$$\mathbf{T} \cdot \mathbf{r} = \lambda\mathbf{r}.$$

For since λ, as we saw above, satisfies the equation of the third degree and must therefore have one of the values λ_1, λ_2, λ_3, it follows, for example, when $\lambda = \lambda_1$, from

$$\begin{aligned}\mathbf{T} \cdot \mathbf{r} &= \lambda_1\mathbf{i}(\mathbf{r} \cdot \mathbf{i}) + \lambda_2\mathbf{j}(\mathbf{r} \cdot \mathbf{j}) + \lambda_3\mathbf{k}(\mathbf{r} \cdot \mathbf{k}) \\ &= \lambda_1\mathbf{r} = \lambda_1[\mathbf{i}(\mathbf{r} \cdot \mathbf{i}) + \mathbf{j}(\mathbf{r} \cdot \mathbf{j}) + \mathbf{k}(\mathbf{r} \cdot \mathbf{k})],\end{aligned}$$

that

$$(\lambda_2 - \lambda_1)\mathbf{j}(\mathbf{r} \cdot \mathbf{j}) + (\lambda_3 - \lambda_1)\mathbf{k}(\mathbf{r} \cdot \mathbf{k}) = 0.$$

Since $\mathbf{j}$ and $\mathbf{k}$ are independent of each other, this equation can be satisfied only if $\mathbf{r} \cdot \mathbf{j}$ and $\mathbf{r} \cdot \mathbf{k}$ are simultaneously zero, that is, if $\mathbf{r}$ has the same or the opposite direction to $\mathbf{i}$.

Similarly for $\lambda = \lambda_2$ or $\lambda = \lambda_3$ we conclude that $\mathbf{r}$ must have the same or the opposite direction to $\mathbf{j}$ or to $\mathbf{k}$; in other words, that only for the corresponding three diameters can the relation

$$\mathbf{T} \cdot \mathbf{r} = \lambda \mathbf{r}$$

be satisfied.

§ 6. Combination of Tensors

Let $\mathbf{T}$ be a transformation which changes $\mathbf{r}$ into $\mathbf{r}'$, and $\mathbf{T}'$ one for which $\mathbf{r}'$ passes into $\mathbf{r}''$, i.e.

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r}; \; \mathbf{r}'' = \mathbf{T}' \cdot \mathbf{r}',$$

then by $\mathbf{T}' \cdot \mathbf{T}$ we mean the transformation which changes $\mathbf{r}$ into $\mathbf{r}''$:

$$\mathbf{r}'' = \mathbf{T}' \cdot \mathbf{r}' = \mathbf{T}' \cdot (\mathbf{T} \cdot \mathbf{r}) = \mathbf{T}'' \cdot \mathbf{r}.$$

Now if

$$\mathbf{T} = \mathbf{ea} + \mathbf{fb} + \mathbf{gc},$$
$$\mathbf{T}' = \mathbf{e}'\mathbf{a}' + \mathbf{f}'\mathbf{b}' + \mathbf{g}'\mathbf{c}',$$

then

$$\mathbf{r}'' = \mathbf{T}' \cdot \mathbf{r}' = \mathbf{e}'(\mathbf{a}' \cdot \mathbf{T} \cdot \mathbf{r}) + \mathbf{f}'(\mathbf{b}' \cdot \mathbf{T} \cdot \mathbf{r}) + \mathbf{g}'(\mathbf{c}' \cdot \mathbf{T} \cdot \mathbf{r}).$$

Each of these terms, for example $\mathbf{e}'(\mathbf{a}' \cdot \mathbf{T} \cdot \mathbf{r})$, may be written

$$\mathbf{e}'(\mathbf{a}' \cdot \mathbf{e})(\mathbf{a} \cdot \mathbf{r}) + \mathbf{e}'(\mathbf{a}' \cdot \mathbf{f})(\mathbf{b} \cdot \mathbf{r}) + \mathbf{e}'(\mathbf{a} \cdot \mathbf{g})(\mathbf{c} \cdot \mathbf{r}),$$

or if $\mathbf{r}$ be taken out as in the simplified method of writing $\mathbf{T}$,

$$[\mathbf{e}'(\mathbf{a}' \cdot \mathbf{e})\mathbf{a} + \mathbf{e}'(\mathbf{a}' \cdot \mathbf{f})\mathbf{b} + \mathbf{e}'(\mathbf{a}' \cdot \mathbf{g})\mathbf{c}] \cdot \mathbf{r}.$$

Each of the three terms gives rise to three such portions so that in all there are nine terms. Each term consists of four vectors of which the two central ones are multiplied together scalarly. We may imagine the nine terms constructed by expanding the product

$$\mathbf{T}' \cdot \mathbf{T} = (\mathbf{e}'\mathbf{a}' + \mathbf{f}'\mathbf{b}' + \mathbf{g}'\mathbf{c}')(\mathbf{ea} + \mathbf{fb} + \mathbf{gc})$$

according to the distributive law, associating each term of the first factor with each term of the second factor without any alteration of the sequence of the vectors, and then uniting as a scalar product the two middle terms in each

product of the four vectors. This scalar product may then be united with the previous or with the succeeding vector to a new vector so that each term is then composed only of two vectors of which the second, for the sake of brevity, again stands for the scalar product with an uninserted vector, to be filled by the vector **r** which is to be transformed.

In this manner $\mathbf{T}' . \mathbf{T}$ again assumes the form of the sum of products of two vectors just like $\mathbf{T}$ and $\mathbf{T}'$ themselves. This sum may again be reduced to three terms in the manner already described, by expressing the second factor in terms of any three mutually independent vectors, and collecting together as one term according to the distributive law all the terms which possess the same second factor; or alternatively by expressing the first factor of each term in this manner and collecting the terms together which have the same first factor.

The transformation $\mathbf{T}' . \mathbf{T}$ must be carefully distinguished from the transformation $\mathbf{T} . \mathbf{T}'$ by which $\mathbf{r}$ is initially transformed into $\mathbf{T}' . \mathbf{r}$ and this vector then changed by $\mathbf{T}$ into $\mathbf{T} . (\mathbf{T}' . \mathbf{r})$. Only in very particular circumstances can $\mathbf{T}' . \mathbf{T}$ be equal to $\mathbf{T} . \mathbf{T}'$.

When $\mathbf{T}$ and $\mathbf{T}'$ are expressed as the sums of more than three products of two vectors at a time, $\mathbf{T}' . \mathbf{T}$ is also found in the same manner as with the sums of three products by removing the brackets according to the distributive law and again in each product of four vectors uniting the central pair into a scalar product.

By the sum of two transformations

$$\mathbf{T}_1 + \mathbf{T}_2$$

we imply the transformation which changes $\mathbf{r}$ into

$$\mathbf{T}_1 . \mathbf{r} + \mathbf{T}_2 . \mathbf{r}.$$

The expression for it can be constructed merely by adding the expressions for $\mathbf{T}_1$ and $\mathbf{T}_2$. From the validity of the distributive law we then obtain directly

$$\mathbf{T} . (\mathbf{T}_1 + \mathbf{T}_2) = \mathbf{T} . \mathbf{T}_1 + \mathbf{T} . \mathbf{T}_2$$

and

$$(\mathbf{T}_1 + \mathbf{T}_2) . \mathbf{T} = \mathbf{T}_1 . \mathbf{T} + \mathbf{T}_2 . \mathbf{T}.$$

By these rules of operation the conception and the association of transformations as a species of multiplication becomes justified.

If a transformation $\mathbf{T}$ is associated with its reciprocal $\mathbf{T}^*$, we obtain, after the manner described above, for $\mathbf{T} . \mathbf{T}^*$ or for $\mathbf{T}^* . \mathbf{T}$ an expression of the same form as that for $\mathbf{T}$. It does not, however, correspond in actual fact to any transformation, for $\mathbf{T}$ and $\mathbf{T}^*$ in association eliminate each other, the one transformation reverses the other, so that an arbitrary vector is transformed into itself. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three arbitrary and mutually independent vectors, and $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ those reciprocal to them, then $\mathbf{T} . \mathbf{T}^*$ and $\mathbf{T}^* . \mathbf{T}$ can be thrown into the form

$$\mathbf{T} . \mathbf{T}^* = \mathbf{T}^* . \mathbf{T} = \mathbf{aa}^* + \mathbf{bb}^* + \mathbf{cc}^* ;$$

for

$$\begin{aligned} &(\mathbf{aa}^* + \mathbf{bb}^* + \mathbf{cc}^*) . \mathbf{r} \\ &= \mathbf{a}(\mathbf{a}^* . \mathbf{r}) + \mathbf{b}(\mathbf{b}^* . \mathbf{r}) + \mathbf{c}(\mathbf{c}^* . \mathbf{r}) = \mathbf{r}. \end{aligned}$$

Since we regard

$$(\mathbf{aa}^* + \mathbf{bb}^* + \mathbf{cc}^*) . \mathbf{r}$$

as a product it is justifiable to represent $\mathbf{aa}^* + \mathbf{bb}^* + \mathbf{cc}^*$ as unity, because $\mathbf{r}$ does not alter on multiplication by this factor.

Hence we may write:

$$\mathbf{T} . \mathbf{T}^* = 1, \text{ or } \mathbf{T}^* = \mathbf{T}^{-1}.$$

§ 7. Resolution into Rotational Tensor and Self-Conjugate Tensor

The two classes of transformation which have received special consideration above, rotational and self-conjugate transformation, provide in combination every arbitrary transformation. Let $\mathbf{T}$ be an arbitrary transformation, then the sphere which is derived when we set

$$\mathbf{r} . \mathbf{r} = 1,$$

and $\mathbf{r}$ is drawn from O, by the transformation, is changed into a surface of the second degree the points of which are

derived by setting off from O a series of vectors given by

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r}.$$

Two mutually perpendicular diameters of the sphere must be transformed into two conjugate diameters of the surface of the second degree. For if at the ends of the diameter of the sphere the tangent planes to the latter are drawn, these must transform into tangent planes of the surface of the second degree. Consequently the other diameter of the sphere, which is parallel to the tangential plane at the end of the perpendicular diameter, must be transformed into a diameter of the surface of the second degree, parallel to the transformed tangent planes.

This may be expressed in the language of vector analysis in the following manner:

From

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r}$$

it follows that

$$d\mathbf{r}' = \mathbf{T} \cdot d\mathbf{r},$$

or

$$\mathbf{r} = \mathbf{T}^{-1} \cdot \mathbf{r}',$$
$$d\mathbf{r} = \mathbf{T}^{-1} \cdot d\mathbf{r}'.$$

The equation of the surface of the second degree is derived from the equation to the sphere by expressing $\mathbf{r}$ in terms of $\mathbf{r}'$, and inserting it into the equation to the sphere:

$$\mathbf{r} \cdot \mathbf{r} = (\mathbf{T}^{-1} \cdot \mathbf{r}') \cdot (\mathbf{T}^{-1} \cdot \mathbf{r}') = 1.$$

In passing from a point of the surface of the second degree $\mathbf{r}'$ to a neighbouring point by altering $\mathbf{r}'$ by an amount $d\mathbf{r}'$, we have that $\mathbf{r}'$ and $d\mathbf{r}'$ are parallel to conjugate diameters.

The condition for conjugate diameters is consequently found by the differentiation of:

$$(\mathbf{T}^{-1} \cdot \mathbf{r}') \cdot (\mathbf{T}^{-1} \cdot \mathbf{r}') = 1.$$

The condition is therefore:

$$2(\mathbf{T}^{-1} \cdot \mathbf{r}') \cdot (\mathbf{T}^{-1} \cdot d\mathbf{r}') = 0.$$

But this is, on the other hand, simply the condition

$$2\mathbf{r} \cdot d\mathbf{r} = 0,$$

i.e. that the corresponding untransformed vectors shall be perpendicular.

The three principal axes of the surface of the second degree, being diameters conjugate to each other, must be derived from three mutually perpendicular diameters of the sphere. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be a right-handed system of three mutually perpendicular vectors of length unity coinciding with these three diameters of the sphere, and $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ three similar vectors coinciding with the principal axes of the surface of the second order. Moreover, let

$$\mathbf{a} = \mathbf{T} \cdot \mathbf{i}, \ \mathbf{b} = \mathbf{T} \cdot \mathbf{j}, \ \mathbf{c} = \mathbf{T} \cdot \mathbf{k},$$

then $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are parallel to the vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, so that we may put

$$\mathbf{a} = \lambda_1\mathbf{i}', \ \mathbf{b} = \lambda_2\mathbf{j}', \ \mathbf{c} = \lambda_3\mathbf{k}'.$$

Now let us construct two transformations for which the first $\mathbf{T}_1$ changes the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, and the second $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$ into $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

$$\begin{aligned} \mathbf{T}_1 &= \mathbf{i}'\mathbf{i} + \mathbf{j}'\mathbf{j} + \mathbf{k}'\mathbf{k} \\ \mathbf{T}_2 &= \mathbf{a}\mathbf{i}' + \mathbf{b}\mathbf{j}' + \mathbf{c}\mathbf{k}' \\ &= \lambda_1\mathbf{i}'\mathbf{i}' + \lambda_2\mathbf{j}'\mathbf{j}' + \lambda_3\mathbf{k}'\mathbf{k}' \\ &= \mathbf{i}'\mathbf{a} + \mathbf{j}'\mathbf{b} + \mathbf{k}'\mathbf{c}. \end{aligned}$$

The first transformation is a rotation, while the second is a self-conjugate transformation. On combining them we derive the transformation $\mathbf{T}$ which changes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$\begin{aligned} \mathbf{T}_2 \cdot \mathbf{T}_1 &= (\mathbf{a}\mathbf{i}' + \mathbf{b}\mathbf{j}' + \mathbf{c}\mathbf{k}')(\mathbf{i}'\mathbf{i} + \mathbf{j}'\mathbf{j} + \mathbf{k}'\mathbf{k}) \\ &= \mathbf{a}\mathbf{i} + \mathbf{b}\mathbf{j} + \mathbf{c}\mathbf{k}. \end{aligned}$$

Alternatively we may take a self-conjugate transformation first which changes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into

$$\mathbf{e} = \lambda_1\mathbf{i}, \ \mathbf{f} = \lambda_2\mathbf{j}, \ \mathbf{g} = \lambda_3\mathbf{k}$$

viz.

$$\mathbf{e}\mathbf{i} + \mathbf{f}\mathbf{j} + \mathbf{g}\mathbf{k}$$

and then a rotation which transforms **e**, **f**, **g** into **a**, **b**, **c**, viz.

$$\mathbf{ae}^* + \mathbf{bf}^* + \mathbf{cg}^*.$$

Then on combination we again arrive at the given transformation **T**:

$$(\mathbf{ae}^* + \mathbf{bf}^* + \mathbf{cg}^*)(\mathbf{ei} + \mathbf{fj} + \mathbf{gk})$$
$$= \mathbf{ai} + \mathbf{bj} + \mathbf{ck}.$$

The rotation is the same in both cases.

For

$$\mathbf{e}^* = \frac{\mathbf{f} \times \mathbf{g}}{\mathbf{efg}} = \frac{1}{\lambda_1}\mathbf{i}\,;\ \mathbf{f}^* = \frac{1}{\lambda_2}\mathbf{j}\,;\ \mathbf{g}^* = \frac{1}{\lambda_3}\mathbf{k},$$

and consequently

$$\mathbf{ae}^* + \mathbf{bf}^* + \mathbf{cg}^* = \mathbf{i'i} + \mathbf{j'j} + \mathbf{k'k}.$$

But the two self-conjugate transformations are not equal. In the one an extension or a contraction takes place in the directions **i′**, **j′**, **k′**, in the other in the directions **i**, **j**, **k**.

§ 8. The Coefficients and Units of a Tensor

If a transformation is given in definite numerical form then there must be specified in numerical form how a given triple set of vectors is transformed into another definite vector triple.

If, for example, we have the transformation

$$\mathbf{T} = \mathbf{ea} + \mathbf{fb} + \mathbf{gc},$$

by which the vectors **a***, **b***, **c*** change into **e**, **f**, **g**, then we suppose the nine coefficients given, by means of which the vectors **e**, **f**, **g** are numerically derived from **a**, **b**, **c**; thus

$$\mathbf{e} = a_{11}\mathbf{a} + a_{12}\mathbf{b} + a_{13}\mathbf{c},$$
$$\mathbf{f} = a_{21}\mathbf{a} + a_{22}\mathbf{b} + a_{23}\mathbf{c},$$
$$\mathbf{g} = a_{31}\mathbf{a} + a_{32}\mathbf{b} + a_{33}\mathbf{c}.$$

T may then also be written:

$$a_{11}\mathbf{aa} + a_{22}\mathbf{bb} + a_{33}\mathbf{cc} + a_{23}\mathbf{cb} + a_{32}\mathbf{bc}$$
$$+ a_{31}\mathbf{ac} + a_{13}\mathbf{ca} + a_{12}\mathbf{ba} + a_{21}\mathbf{ab}.$$

In the complete expression there would be written in place of the last factor of each term the undetermined expression **a . l**, in place of **a**, instead of **b** there would be **b . l**, and instead of **c**, **c . l**, where **l** is to be replaced by **r** in the transformation **T . r**.

The conjugate transformation

$$\overline{\mathbf{T}} = \mathbf{ae} + \mathbf{bf} + \mathbf{cg}$$

may in analogous manner be written

$$\alpha_{11}\mathbf{aa} + \alpha_{22}\mathbf{bb} + \alpha_{33}\mathbf{cc} + \alpha_{32}\mathbf{cb} + \alpha_{23}\mathbf{bc} + \alpha_{13}\mathbf{ac} + \alpha_{31}\mathbf{ca} + \alpha_{21}\mathbf{ba} + \alpha_{12}\mathbf{ab},$$

by inserting the expressions for **e**, **f**, **g**. That is to say, the expression for **T** transforms into that for $\overline{\mathbf{T}}$ if the two vectors are interchanged in each expression, or what amounts to the same thing, if the two indices of the coefficients are interchanged in each term.

A self-conjugate transformation is accordingly characterised by the equalities

$$\alpha_{32} = \alpha_{23},\ \alpha_{13} = \alpha_{31},\ \alpha_{21} = \alpha_{12}.$$

The nine coefficients

$$\begin{matrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{matrix}$$

we term the co-ordinates or the coefficients of the transformation or of the tensor **T** in relation to the nine units:

aa, ab, ac
ba, bb, bc
ca, cb, cc.

By an appropriate choice of these nine coefficients the vectors **e**, **f**, **g** may be made equal to any three given vectors, and hence express every affine transformation. In order to express an affine transformation the values of the nine coefficients must satisfy the single condition, that their determinant does not vanish. For, as we have already seen, the determinant represents the ratio of the two volumes **efg**

and $\mathbf{a^*b^*c^*}$ which do not vanish since both **a, b, c** and **e, f, g** are assumed independent of each other.

§ 9. Tensors of Fewer than three Terms

We propose to retain the assumption that **a, b, c** are independent of each other but admit that **e, f, g** are dependent on each other in such manner in the first place that two of them, **e** and **f** for example, remain independent. The external product **efg** then vanishes and with it the determinant of the nine coefficients. The transformation

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r}$$

may still be effected for every arbitrary vector **r** in the same manner as before but the vectors **r**′ no longer represent all possible vectors of space but those only which are capable of derivation numerically from **e** and **f**, that is to say, are parallel to a plane. If **g** be expressed in terms of **e** and **f**, and the terms associated which have **e** as their first factor, and likewise those terms having **f** as their first factor, then **T** takes the form

$$\mathbf{T} = \mathbf{eo} + \mathbf{fp},$$

where **o** and **p** are numerically derived from **a, b, c** in a definite manner. If **o** and **p** are independent, we may add a third vector **q** which is independent of **o** and **p**.

Then **e** and **f** may be derived numerically from **o, p, q** in the form:

$$\mathbf{e} = b_{11}\mathbf{o} + b_{12}\mathbf{p} + b_{13}\mathbf{q}$$
$$\mathbf{f} = b_{21}\mathbf{o} + b_{22}\mathbf{p} + b_{23}\mathbf{q}.$$

Such a transformation

$$\mathbf{T} = \mathbf{eo} + \mathbf{fp}$$

is already characterised by six coefficients:

$$\begin{matrix} b_{11}, & b_{12}, & b_{13}, \\ b_{21}, & b_{22}, & b_{23}. \end{matrix}$$

We term this its coefficients relative to the system **o, p, q.**

If **e, f, g** depend more closely on each other, so that they may be derived numerically from some one of them, for

example from **e**, then all the terms of **T** may be grouped together in a single one of the form:

$$\mathbf{T} = \mathbf{eo}.$$

The transformation makes

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r} = \mathbf{e}(\mathbf{o} \cdot \mathbf{r});$$

all vectors **r** into vectors **r**′ which are numerically derivable from **e**. If to **o** we add two other vectors **p** and **q** such that **o**, **p**, **q** are mutually independent, then **e** can be numerically derived from **o**, **p**, **q**:

$$\mathbf{e} = c_{11}\mathbf{o} + c_{12}\mathbf{p} + c_{13}\mathbf{q}.$$

A transformation $\mathbf{T} = \mathbf{eo}$ is therefore given by the three coefficients

$$c_{11}, \quad c_{12}, \quad c_{13}.$$

These we term its coefficients with reference to **o**, **p**, **q**.

All three types of transformation are included in the form

$$\mathbf{ea} + \mathbf{fb} + \mathbf{gc},$$

if the assumption that **e**, **f**, **g** are mutually independent is discarded. According as two of them remain independent of each other or all are derivable from one, the transformed vectors become parallel to a plane or to a straight line. This last type is, so to speak, the most elementary constituent of the general transformation in so far as the latter may be represented as a sum of transformations of this simple type.

To these transformations there belong also those obtained when the vector **r** is multiplied vectorially by a fixed vector **o**.

Suppose that in addition to **o** two other vectors are chosen such that

$$\mathbf{opq} = 1,$$

then $\quad \mathbf{o}^* = \mathbf{p} \times \mathbf{q},\ \mathbf{p}^* = \mathbf{q} \times \mathbf{o},\ \mathbf{q}^* = \mathbf{o} \times \mathbf{p},$

now $\quad \mathbf{r} = \mathbf{o}(\mathbf{o}^* \cdot \mathbf{r}) + \mathbf{p}(\mathbf{p}^* \cdot \mathbf{r}) + \mathbf{q}(\mathbf{q}^* \cdot \mathbf{r}),$

consequently

$$\mathbf{o} \times \mathbf{r} = \mathbf{q}^*(\mathbf{p}^* \cdot \mathbf{r}) - \mathbf{p}^*(\mathbf{q}^* \cdot \mathbf{r})$$
$$= \mathbf{T} \cdot \mathbf{r},$$

where

$$\mathbf{T} = \mathbf{q}^*\mathbf{p}^* - \mathbf{p}^*\mathbf{q}^*.$$

Conversely it may be shown that each transformation of this form

$$\mathbf{T} = \mathbf{ba} - \mathbf{ab}$$

may be replaced by a vectorial product of the variable vector with a fixed vector $\mathbf{a} \times \mathbf{b}$. For we have:

$$\mathbf{T} \cdot \mathbf{r} = \mathbf{b}(\mathbf{a} \cdot \mathbf{r}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{r}).$$

If a vector $\mathbf{c}$ be determined so that $\mathbf{abc} = 1$ and we put

$$\mathbf{a}^* = \mathbf{b} \times \mathbf{c}; \ \mathbf{b}^* = \mathbf{c} \times \mathbf{a}; \ \mathbf{c}^* = \mathbf{a} \times \mathbf{b},$$

then

$$\mathbf{r} = \mathbf{a}^*(\mathbf{a} \cdot \mathbf{r}) + \mathbf{b}^*(\mathbf{b} \cdot \mathbf{r}) + \mathbf{c}^*(\mathbf{c} \cdot \mathbf{r})$$

and therefore

$$\mathbf{c}^* \times \mathbf{r} = \mathbf{b}(\mathbf{a} \cdot \mathbf{r}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{r}) = \mathbf{T} \cdot \mathbf{r},$$

a result that might have been derived directly from the formula

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{r} = \mathbf{b}(\mathbf{a} \cdot \mathbf{r}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{r})$$

found in Chap. I, § 15.

The conjugate transformation $\overline{\mathbf{T}}$ is here equal to $-\mathbf{T}$ so that the sum $\mathbf{T} + \overline{\mathbf{T}}$, which, as has been seen, represented in general a self-conjugate tensor and thus provided a scalar function $2\mathbf{Tr} \cdot \mathbf{r}$, in this case vanishes.

If any transformation

$$\mathbf{T} = \mathbf{ea} + \mathbf{fb} + \mathbf{gc}$$

is the reverse of its conjugate

$$\overline{\mathbf{T}} = \mathbf{ae} + \mathbf{bf} + \mathbf{cg},$$

then

$$\mathbf{T} = \tfrac{1}{2}(\mathbf{T} - \overline{\mathbf{T}})$$
$$= \tfrac{1}{2}(\mathbf{ea} - \mathbf{ae}) + \tfrac{1}{2}(\mathbf{fb} - \mathbf{bf}) + \tfrac{1}{2}(\mathbf{gc} - \mathbf{cg}),$$

i.e. the transformation $\mathbf{T}$ is a sum of three transformations

of the type just considered which may be replaced by vectorial multiplication by a fixed vector. The transformation $\mathbf{T} \cdot \mathbf{r}$ has accordingly the same significance as

$$\tfrac{1}{2}(\mathbf{a} \times \mathbf{e} + \mathbf{b} \times \mathbf{f} + \mathbf{c} \times \mathbf{g}) \times \mathbf{r},$$

or if we put

$$\begin{aligned}&\tfrac{1}{2}(\mathbf{a} \times \mathbf{e} + \mathbf{b} \times \mathbf{f} + \mathbf{c} \times \mathbf{g}) = \mathbf{p} \times \mathbf{q},\\ &\mathbf{T} = \mathbf{qp} - \mathbf{pq}.\end{aligned}$$

Every transformation $\mathbf{T}$, which is the opposite of its conjugate, may therefore be thrown into the form

$$\mathbf{qp} - \mathbf{pq},$$

and $\mathbf{T} \cdot \mathbf{r}$ is simply the vectorial product

$$(\mathbf{p} \times \mathbf{q}) \times \mathbf{r}.$$

Such a transformation is completely given by a vector $\mathbf{p} \times \mathbf{q}$, or if we will, by the associated vectorial area $\mathbf{pq}$, the representation of the vector.

To construct a picture of the transformed vector $\mathbf{T} \cdot \mathbf{r}$ we will suppose the vectorial area placed in the plane of the diagram and also the fixed point O from which $\mathbf{r}$ is drawn. The end point of $\mathbf{r}$ is projected perpendicular to the plane of the drawing, to P. The vector OP $= \mathbf{r}_1$ differs from $\mathbf{r}$ by a vector which is perpendicular to the drawing, i.e. it is parallel to $\mathbf{p} \times \mathbf{q}$.

FIG. 31.

Hence

$$\mathbf{T} \cdot \mathbf{r} = (\mathbf{p} \times \mathbf{q}) \times \mathbf{r} = (\mathbf{p} \times \mathbf{q}) \times \mathbf{r}_1,$$

i.e. all vectors $\mathbf{r}$ drawn out from O to any point on the same straight line perpendicular to the plane are transformed into the same vector $\mathbf{T} \cdot \mathbf{r}$. This is at right angles to OP on the side corresponding to the *sense* of the vectorial area $\mathbf{pq}$. If the numerical value of the vectorial area equals unity, the length of $\mathbf{T} \cdot \mathbf{r}$ equals that of $\mathbf{r}_1$, if it is greater or less than unity, the length of $\mathbf{T} \cdot \mathbf{r}$ is in the same proportion greater or less than the length of $\mathbf{r}_1$. Drawn from P, $\mathbf{T} \cdot \mathbf{r}$ leads to the point P′. Then tan $\angle$ P′OP is equal to the numerical value of the vectorial area $\mathbf{pq}$ (fig. 31).

§ 10. Symmetric and Asymmetric Tensors

If
$$\mathbf{T} = \mathbf{ea} + \mathbf{fb} + \mathbf{gc}$$
is an arbitrary transformation, and if half the sum and half the difference of $\mathbf{T}$ and its conjugate $\overline{\mathbf{T}}$ are taken, then
$$\mathbf{T}_1 = \frac{\mathbf{T} + \overline{\mathbf{T}}}{2}; \ \mathbf{T}_2 = \frac{\mathbf{T} - \overline{\mathbf{T}}}{2},$$
and
$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2,$$
and on the one hand $\mathbf{T}_1 = \overline{\mathbf{T}}_1$, and on the other $\mathbf{T}_2 = -\overline{\mathbf{T}}_2$. For we obtain $\overline{\mathbf{T}}_1$ and $\overline{\mathbf{T}}_2$ if the vectors are interchanged in all the terms, so that $\mathbf{T}$ transforms into $\overline{\mathbf{T}}$. Hence
$$\overline{\mathbf{T}}_1 = \frac{\overline{\mathbf{T}} + \mathbf{T}}{2}; \ \overline{\mathbf{T}}_2 = \frac{\overline{\mathbf{T}} - \mathbf{T}}{2}.$$

In other words, every arbitrary transformation may be presented as the sum of two transformations of which the one is its own self-conjugate and the other is its own conjugate reversed.

Hence $\mathbf{T} \cdot \mathbf{r}$ is composed of $\mathbf{T}_1 \cdot \mathbf{r} + \mathbf{T}_2 \cdot \mathbf{r}$.

$\mathbf{T}_1 \cdot \mathbf{r}$ is, as we saw above, the half gradient of the scalar function $\mathbf{T}_1 \cdot \mathbf{r} \cdot \mathbf{r}$, and $\mathbf{T}_2 \cdot \mathbf{r}$ is the vectorial product:
$$\tfrac{1}{2}(\mathbf{a} \times \mathbf{e} + \mathbf{b} \times \mathbf{f} + \mathbf{c} \times \mathbf{g}) \times \mathbf{r}.$$

This brings out the geometrical interpretation of the vectorial area
$$\mathbf{ae} + \mathbf{bf} + \mathbf{cg}$$
which we have discussed to some extent already. One half its representation is the vector whose vectorial product with $\mathbf{r}$ gives the transformed vector $\mathbf{T}_2 \cdot \mathbf{r}$. Moreover, the geometrical interpretation of
$$\mathbf{e} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{f} + \mathbf{c} \cdot \mathbf{g}$$
becomes evident. Since $\mathbf{T}_1$ is its own conjugate this tensor, as we have seen already, may be thrown into the form
$$\mathbf{T}_1 = \lambda_1 \mathbf{ii} + \lambda_2 \mathbf{jj} + \lambda_3 \mathbf{kk}.$$

On the other hand,

$$\mathbf{T}_1 = \tfrac{1}{2}(\mathbf{ea} + \mathbf{ae} + \mathbf{fb} + \mathbf{bf} + \mathbf{gc} + \mathbf{cg})$$

and therefore from § 1, Chap. III,

$$\begin{aligned}\mathbf{e}\cdot\mathbf{a} + \mathbf{f}\cdot\mathbf{b} + \mathbf{g}\cdot\mathbf{c} &= \lambda_1\mathbf{i}\cdot\mathbf{i} + \lambda_2\mathbf{j}\cdot\mathbf{j} + \lambda_3\mathbf{k}\cdot\mathbf{k}\\ &= \lambda_1 + \lambda_2 + \lambda_3.\end{aligned}$$

Hence $\mathbf{e}\cdot\mathbf{a} + \mathbf{f}\cdot\mathbf{b} + \mathbf{g}\cdot\mathbf{c}$ equals the sum of the three numbers λ_1, λ_2, λ_3 which specify how the volume is expanded or compressed in the three mutually perpendicular directions and, if they negative, mirrored in addition, in order to apply the transformation $\mathbf{T}_1$.

§ 11. Reciprocal Tensors

The transformation reciprocal to

$$\mathbf{T} = \mathbf{ea} + \mathbf{fb} + \mathbf{gc}$$

changes the vectors **e**, **f**, **g** back into $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$, the three vectors reciprocal to **a**, **b**, **c**. Accordingly, it may be written in the form

$$\mathbf{T}^{-1} = \mathbf{a}^*\mathbf{e}^* + \mathbf{b}^*\mathbf{f}^* + \mathbf{c}^*\mathbf{g}^*.$$

This naturally assumes that **e**, **f**, **g** are independent of each other. For if they were dependent, the external product **efg** would be zero and the reciprocal vectors

$$\mathbf{e}^* = \frac{\mathbf{f}\times\mathbf{g}}{\mathbf{efg}};\ \mathbf{f}^* = \frac{\mathbf{g}\times\mathbf{e}}{\mathbf{efg}};\ \mathbf{g}^* = \frac{\mathbf{e}\times\mathbf{f}}{\mathbf{efg}}$$

could not be constructed.

The numerical relations between the vectors $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ and the vectors $\mathbf{e}^*$, $\mathbf{f}^*$, $\mathbf{g}^*$ may be expressed by means of the same nine coefficients $a_{\alpha\beta}$ in terms of which **e**, **f**, **g** were derived numerically from **a**, **b**, **c**.

For, from

$$\mathbf{e} = a_{11}\mathbf{a} + a_{12}\mathbf{b} + a_{13}\mathbf{c}, \text{ etc.},$$

it follows that

$$\mathbf{e}\cdot\mathbf{a}^* = a_{11},\ \mathbf{e}\cdot\mathbf{b}^* = a_{12},\ \mathbf{e}\cdot\mathbf{c}^* = a_{13}, \text{ etc.}$$

Hence:

$$\begin{aligned}\mathbf{a}^* &= \alpha_{11}\mathbf{e}^* + \alpha_{21}\mathbf{f}^* + \alpha_{31}\mathbf{g}^*\\ \mathbf{b}^* &= \alpha_{12}\mathbf{e}^* + \alpha_{22}\mathbf{f}^* + \alpha_{32}\mathbf{g}^*\\ \mathbf{c}^* &= \alpha_{13}\mathbf{e}^* + \alpha_{23}\mathbf{f}^* + \alpha_{33}\mathbf{g}^*.\end{aligned}$$

In the scheme for the nine coefficients

$$\begin{matrix}\alpha_{11} & \alpha_{21} & \alpha_{31}\\ \alpha_{12} & \alpha_{22} & \alpha_{32}\\ \alpha_{13} & \alpha_{23} & \alpha_{33}\end{matrix}$$

the columns are merely interchanged with the rows and the coefficients for the reciprocal transformation relate not to the system **a, b, c** to which the coefficients $\alpha_{\alpha\beta}$ for **T** related, but to the system **e*, f*, g.** If we wish to express $\mathbf{T}^{-1}$ referred to **a*, b*, c*** we must find the values of **e*, f*, g*** in terms of **a*, b*, c***, i.e. we have to conduct the vectorial calculations

$$\mathbf{e}^* = \frac{\mathbf{f} \times \mathbf{g}}{\mathbf{efg}}; \ \mathbf{f}^* = \frac{\mathbf{g} \times \mathbf{e}}{\mathbf{efg}}; \ \mathbf{g}^* = \frac{\mathbf{e} \times \mathbf{f}}{\mathbf{efg}},$$

by inserting the expressions for **e, f, g** in terms of **a*, b*, c*.** This gives:

$$\begin{aligned}\mathbf{e}^* &= \alpha^*_{11}\mathbf{a}^* + \alpha^*_{12}\mathbf{b}^* + \alpha^*_{13}\mathbf{c}^*\\ \mathbf{f}^* &= \alpha^*_{21}\mathbf{a}^* + \alpha^*_{22}\mathbf{b}^* + \alpha^*_{32}\mathbf{c}^*\\ \mathbf{g}^* &= \alpha^*_{31}\mathbf{a}^* + \alpha^*_{32}\mathbf{b}^* + \alpha^*_{33}\mathbf{c}^*.\end{aligned}$$

This is the reversion of the vector equations already written down, by means of which **a*, b*, c*** were numerically derived from **e*, f*, g*.**

The same coefficients $\alpha^*_{\alpha\beta}$ give the reversion of the equations

$$\mathbf{e} = \alpha_{11}\mathbf{a} + \alpha_{12}\mathbf{b} + \alpha_{13}\mathbf{c}, \text{ etc.},$$

if the indices are merely interchanged:

$$\begin{aligned}\mathbf{a} &= \alpha^*_{11}\mathbf{e} + \alpha^*_{21}\mathbf{f} + \alpha^*_{31}\mathbf{g}\\ \mathbf{b} &= \alpha^*_{12}\mathbf{e} + \alpha^*_{22}\mathbf{f} + \alpha^*_{32}\mathbf{g}\\ \mathbf{c} &= \alpha^*_{13}\mathbf{e} + \alpha^*_{23}\mathbf{f} + \alpha^*_{33}\mathbf{g}.\end{aligned}$$

For in an analogous manner by which we passed from **T** to $\mathbf{T}^{-1}$ so we arrive at the reciprocal transformation **T** from

the transformation $\mathbf{T}^{-1}$ which was referred to $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$, that is to say, which was supposed given by the coefficients $a^*_{\alpha\beta}$. Thus $\mathbf{T}$ is determined by the same coefficients $a^*_{\alpha\beta}$ except that the indices must be interchanged and they are to relate not to the system $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$, but to $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$.

§ 12. The Tensor Idea

A relation between two variable vectors $\mathbf{r}$ and $\mathbf{r}'$ such as is expressed by

$$\mathbf{r}' = \mathbf{T} \cdot \mathbf{r}$$

arises not merely in the case of affine transformations of space, but also with many other mathematical and physical considerations which have nothing essentially to do with space transformations. We have already encountered two such cases. In the first instance $\mathbf{r}'$ was represented by the gradient of a definite scalar function of position $f(\mathbf{r})$, while in the second case $\mathbf{r}'$ was derived as the vectorial product of a constant vector with $\mathbf{r}$. As a physical illustration, consider the stresses operating in the various planes passing through a point in an elastic body under the influence of any forces. If the body be supposed cut in any one of the planes and the one portion removed, then we must imagine certain forces operating on the elements of the intersecting plane if everything is to remain in the previous state of equilibrium. At the point under consideration we may imagine an indefinitely small vectorial area lying in the section and indicating by its *sense* on which side the fixed body lies, for example, by postulating that it lies on the positive side of the vectorial area. Each such vectorial area at the point considered corresponds to a certain force which must be applied there to maintain the equilibrium.

Had we retained the portion removed, and removed the portion retained, the vectorial area with the opposite *sense* would have had to be taken in order that the portion now retained should again lie on its positive side; in which case the force would equal the original one but reversed in sign. The description of the state of stress at the position considered consists in a statement of the relation between

vectorial area and force. We may imagine the vectorial area replaced by its *representation* so that we have to deal for a given state of stress with a relation between two vectors **r** and **r**′, one of which corresponds to an arbitrary indefinitely small vectorial area and the other to the correspondingly small force. In the theory of elasticity it is shown that to a first approximation for not too great deformations of the elastic body the relation between **r** and **r**′ is of the type we have met in affine transformations. We shall encounter yet other illustrations of such relations between variable vectors. These applications to the state of elastic stress has led to the use of the word "tensor" for the idea involved in the passage from **r**′ to **r**.

Originally the word referred specifically to the state of stress (*tendere* = to stretch, *tensio* = tension); now-a-days, however, it has acquired the colourless abstract significance that it merely expresses the relation of one vector to another, whether or not it refers to an actual state of stress. From now onwards we will use the word in this same neutral sense and therefore speak of a tensor

$$\mathbf{T} = \mathbf{ea} + \mathbf{fb} + \mathbf{gc},$$

of the conjugate tensor

$$\bar{\mathbf{T}} = \mathbf{ae} + \mathbf{bf} + \mathbf{cg},$$

of the reciprocal tensor

$$\mathbf{T}^{-1} = \mathbf{a}^*\mathbf{e}^* + \mathbf{b}^*\mathbf{f}^* + \mathbf{c}^*\mathbf{g}^*,$$

of the tensor which arises by the addition of two tensors $\mathbf{T}_1 . \mathbf{T}_2$, and by the product of two tensors, etc.

Immediately it is desired to determine a tensor numerically, it must be referred to three mutually independent vectors **a**, **b**, **c**, and, as we have already seen, it may then be written in the form

$$a_{11}\mathbf{aa} + a_{22}\mathbf{bb} + a_{33}\mathbf{cc} + a_{23}\mathbf{cb} + a_{32}\mathbf{bc} + a_{31}\mathbf{ac} + a_{13}\mathbf{ca} + a_{21}\mathbf{ab} + a_{12}\mathbf{ba}.$$

It is in fact derived numerically from the nine tensors

$$\mathbf{aa},\ \mathbf{bb},\ \mathbf{cc},\ \mathbf{bc},\ \mathbf{cb},\ \mathbf{ca},\ \mathbf{ac},\ \mathbf{ab},\ \mathbf{ba}$$

by means of the nine numbers $a_{\alpha\beta}$.

We term the nine tensors **aa, bb, cc, bc**, etc., the unit tensors of the system **a, b, c**, and $a_{\alpha\beta}$ the coefficients of the tensor with reference to these units where the values 1, 2, 3 of the first index correspond to the second factors **a, b, c** and the values 1, 2, 3 of the second index to the first factors.

Just as a vector is derived numerically from three vectors **a, b, c** by means of its coefficients, so a tensor is derived from the nine unit tensors of the system.

If $a_{\alpha\beta} = a_{\beta\alpha}$, then, as we saw already, the tensor is self-conjugate. We will term this a "symmetric tensor." If the coefficients of **r** referred to **a***, **b***, **c*** be denoted by x, y, z, then for a symmetric tensor

$$\mathbf{T}\cdot\mathbf{r}\cdot\mathbf{r} = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy,$$

and the relation

$$\mathbf{r}' = \mathbf{T}\cdot\mathbf{r},$$

as we have already found, may be represented by setting **r′** equal to the gradient of the scalar function of position

$$\tfrac{1}{2}\mathbf{T}\cdot\mathbf{r}\cdot\mathbf{r}.$$

The vector field of the gradient enables us in this case to obtain a geometrical picture of the relation between **r′** and **r**. We associate with each vector **r** the gradient at that point of space at which the vector terminates. Now to each symmetrical tensor there corresponds a quadratic form in three variables x, y, z:

$$\mathbf{T}\cdot\mathbf{r}\cdot\mathbf{r} = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy,$$

where x, y, z are the coefficients of **r** with reference to **a***, **b***, **c***.

And conversely to every such quadratic form for x, y, z where x, y, z are coefficients referred to three arbitrary and mutually independent vectors **a***, **b***, **c***, there corresponds a symmetric tensor:

$$\mathbf{T} = a_{11}\mathbf{aa} + a_{22}\mathbf{bb} + a_{33}\mathbf{cc} + a_{23}(\mathbf{bc} + \mathbf{cb}) + a_{31}(\mathbf{ca} + \mathbf{ac}) + a_{12}(\mathbf{ab} + \mathbf{ba}).$$

The relation between $\mathbf{r}$ and $\mathbf{r}'$, as we saw before, may be presented by constructing the surface of the second order

$$\mathbf{r}' \cdot \mathbf{r} = \mathbf{T} \cdot \mathbf{r} \cdot \mathbf{r} = 1$$

by drawing $\mathbf{r}$ from the point O. For every $\mathbf{r}$, $\mathbf{r}'$ is perpendicular to the surface at the terminal point of $\mathbf{r}$. Hence if drawn from O the end of the vector $\mathbf{r}'$ falls on the perpendicular which can be dropped from O on to the tangential plane at the end of $\mathbf{r}$, and its length equals the reciprocal of the length of the perpendicular. For a different surface of the second degree corresponding to a different constant value of $\mathbf{T} \cdot \mathbf{r} \cdot \mathbf{r}$ we need merely imagine $\mathbf{r}$ and $\mathbf{r}'$ altered in the same proposition. If $\mathbf{r}$ and $\mathbf{r}'$ both increase n-fold $\mathbf{r}' \cdot \mathbf{r}$ becomes n^2 times as great.

The symmetric tensors accordingly provide a picture in a way of the quadratic forms of three variables.

A tensor which is opposite to its conjugate we term an "asymmetric tensor."

For it $\quad a_{11} = a_{22} = a_{33} = 0$

and $\quad a_{23} = -a_{32},\ a_{31} = -a_{13},\ a_{12} = -a_{21},$

as already proved, so that it assumes the form

$$\mathbf{T} = a_{23}(\mathbf{cb} - \mathbf{bc}) + a_{31}(\mathbf{ac} - \mathbf{ca}) + a_{12}(\mathbf{ba} - \mathbf{ab}).$$

If x, y, z are again the coefficients by means of which $\mathbf{r}$ is numerically derived from $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$, then

$$\begin{aligned}
\mathbf{r}' &= \mathbf{T} \cdot \mathbf{r} \\
&= a_{23}(\mathbf{c}y - \mathbf{b}z) + a_{31}(\mathbf{a}z - \mathbf{c}x) + a_{12}(\mathbf{b}x - \mathbf{a}y) \\
&= a_{23}\frac{\mathbf{a}^* \times \mathbf{r}}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*} + a_{31}\frac{\mathbf{b}^* \times \mathbf{r}}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*} + a_{12}\frac{\mathbf{c}^* \times \mathbf{r}}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*} \\
&= \frac{a_{23}\mathbf{a}^* + a_{31}\mathbf{b}^* + a_{12}\mathbf{c}^*}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*} \times \mathbf{r},
\end{aligned}$$

i.e. the vector $\mathbf{r}'$ arises from $\mathbf{r}$ when the constant vector

$$\frac{a_{23}\mathbf{a}^* + a_{31}\mathbf{b}^* + a_{12}\mathbf{c}^*}{\mathbf{a}^*\mathbf{b}^*\mathbf{c}^*}$$

$$= a_{23}(\mathbf{b} \times \mathbf{c}) + a_{31}(\mathbf{c} \times \mathbf{a}) + a_{12}(\mathbf{a} \times \mathbf{b})$$

is multiplied vectorially by **r**. This vector is the representation of the vectorial area:

$$a_{23}\mathbf{bc} + a_{31}\mathbf{ca} + a_{12}\mathbf{ab}.$$

If in the expression for the asymmetric tensor

$$\mathbf{T} = a_{23}(\mathbf{cb} - \mathbf{bc}) + a_{31}(\mathbf{ac} - \mathbf{ca}) + a_{12}(\mathbf{ba} - \mathbf{ab})$$

all the terms are conceived as if they were external products of two vectors, then **T** transforms into the vectorial area

$$\mathbf{P} = 2a_{23}\mathbf{cb} + 2a_{31}\mathbf{ac} + 2a_{12}\mathbf{ba}.$$

The relation between **r**′ and **r** may then be presented in the following fashion: the vectorial product

$$\mathbf{r}' = \mathbf{r} \times | \tfrac{1}{2}\mathbf{P}$$

is constructed. Then every asymmetric tensor corresponds to a definite vectorial area **P** and conversely every vectorial area **P** corresponds to a definite asymmetric tensor which merely represents the passage from a variable vector **r** to the vectorial product of **r** by half the *representation* of **P**. The vectorial area **P** may then be written exactly like the tensor **T**:

$$\mathbf{P} = a_{23}(\mathbf{cb} - \mathbf{bc}) + a_{31}(\mathbf{ac} - \mathbf{ca}) + a_{12}(\mathbf{ba} - \mathbf{ab}),$$

except that here the separate terms represent external products of the vectors **a**, **b**, **c**.

Every arbitrary tensor **T** may be expressed as the sum of a symmetric and an asymmetric tensor:

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2.$$

The symmetric tensor $\mathbf{T}_1$ equals half the sum of **T** and its conjugate $\overline{\mathbf{T}}$, the asymmetric tensor is half the difference of **T** and its conjugate $\overline{\mathbf{T}}$.

For example, let there be given a rotation through an angle ϑ about an axis passing through O. A vector **r** drawn from O changes into a vector **r**′. The transition defines a tensor **T**. Let us resolve this tensor into a symmetric tensor $\mathbf{T}_1$ and an asymmetric tensor $\mathbf{T}_2$. As we have

seen, in a rotation the conjugate tensor equals the reciprocal tensor, and consequently:

$$\mathbf{T}_1 = \frac{\mathbf{T} + \mathbf{T}^{-1}}{2};\ \mathbf{T}_2 = \frac{\mathbf{T} - \mathbf{T}^{-1}}{2}.$$

Imagine the axis of rotation is at right angles to the plane of the drawing (fig. 32). During the rotation a point P in the plane of the drawing moves along the arc of a circle to P′, while during the reciprocal transformation it moves to P″. Every point on a line through P perpendicular to the drawing moves parallel to that plane and remains in the same perpendicular as the moving point in the plane. Similarly with the reciprocal rotation.

If $\mathbf{r}$ is the vector which joins O to the original position of the point then it can be separated into two parts, the vector joining O to P and a constant portion $\mathbf{c}$ perpendicular to the plane of the drawing. Then clearly $\mathbf{T}\,.\,\mathbf{r}$ equals the vector OQ + $\mathbf{c}$, where Q lies midway between P and P′ and $\mathbf{T}_2\,.\,\mathbf{r}$ equals half the vector P″P′, i.e. equals the vector QP′.

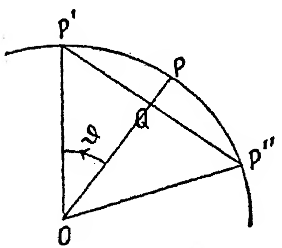

FIG. 32.

The transformation $\mathbf{T}\,.\,\mathbf{r}$ moves all lines parallel to the axis of rotation nearer to that axis so that its distance is diminished in the ratio cos ϑ (a negative value of cos ϑ indicates that the straight line passes to the opposite side of O), it compresses the region symmetrically about the axis of rotation, while with negative cos ϑ an inversion also occurs. The transformation $\mathbf{T}_2\,.\,\mathbf{r}$ adds to $\mathbf{T}\,.\,\mathbf{r}$ the vector QP′ which is the vectorial product of a vector perpendicular to the plane of the drawing of length sin ϑ, with the vector $\mathbf{r}$.

In terms of vector analysis these geometrical considerations would be presented as follows:

We assume $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is a self-reciprocal system, $\mathbf{k}$ being parallel to the axis of rotation. The vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are changed into $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}$ by the rotation, where:

$$\mathbf{i}' = \cos\vartheta\mathbf{i} + \sin\vartheta\mathbf{j}$$
$$\mathbf{j}' = -\sin\vartheta\mathbf{i} + \cos\vartheta\mathbf{j}.$$

Then:

$$\begin{aligned} \mathbf{T} &= \mathbf{i'i} + \mathbf{j'j} + \mathbf{kk} \\ &= \cos\vartheta(\mathbf{ii} + \mathbf{jj}) + \mathbf{kk} + \sin\vartheta(\mathbf{ji} - \mathbf{ij}) \\ \bar{\mathbf{T}} &= \mathbf{ii'} + \mathbf{jj'} + \mathbf{kk} \\ &= \cos\vartheta(\mathbf{ii} + \mathbf{jj}) + \mathbf{kk} - \sin\vartheta(\mathbf{ji} - \mathbf{ij}). \end{aligned}$$

The first part is the symmetric tensor

$$\mathbf{T}_1 = \cos\vartheta(\mathbf{ii} + \mathbf{jj}) + \mathbf{kk},$$

which changes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into $\cos\vartheta\ \mathbf{i}$, $\cos\vartheta\ \mathbf{j}$, $\mathbf{k}$, while the second portion is the asymmetric tensor

$$\mathbf{T}_2 = \sin\vartheta(\mathbf{ji} - \mathbf{ij}),$$

which converts the vector

$$\mathbf{r} \times \mid \tfrac{1}{2}\mathbf{P}$$

into the vector $\mathbf{r}$, where

$$\begin{aligned} \mathbf{P} &= -2\sin\vartheta\ \mathbf{ij}, \\ \mid \tfrac{1}{2}\mathbf{P} &= -\sin\vartheta\ \mathbf{k}, \end{aligned}$$

and therefore

$$\mathbf{r} \times \mid \tfrac{1}{2}\mathbf{P} = \sin\vartheta\ \mathbf{k} \times \mathbf{r}.$$

If the angle of rotation ϑ is small then $1 - \cos\vartheta$ is of the second order in comparison with $\sin\vartheta$. Accordingly in the expression for the rotational tensor we will introduce

$$\cos\vartheta = 1 - 2\sin^2\vartheta/2,$$

and write

$$\mathbf{T} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} + \sin\vartheta(\mathbf{ji} - \mathbf{ij}) - 2\sin^2\frac{\vartheta}{2}(\mathbf{ii} + \mathbf{jj}).$$

If that portion associated with $\sin^2\vartheta/2$ in the formation of $\mathbf{r}' = \mathbf{T} . \mathbf{r}$ can be neglected, then

$$\mathbf{T} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} + \sin\vartheta(\mathbf{ji} - \mathbf{ij}),$$

and having already discarded terms of the second order we may substitute the angle itself for $\sin\vartheta$:

$$\mathbf{T} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} + \vartheta(\mathbf{ji} - \mathbf{ij}).$$

The first portion leaves the vector $\mathbf{r}$ unaffected in constructing $\mathbf{T} . \mathbf{r}$, so that we get

$$\mathbf{T} . \mathbf{r} = \mathbf{r} + \vartheta(\mathbf{j}x - \mathbf{i}y)$$

or

$$\mathbf{T} = \mathrm{I} + \mathbf{T}_2$$

where $\mathbf{T}_2$ represents the indefinitely small asymmetric tensor and I is written for the expression:

$$\mathbf{ii} + \mathbf{jj} + \mathbf{kk} \text{ or } \mathbf{a}^*\mathbf{a} + \mathbf{b}^*\mathbf{b} + \mathbf{c}^*\mathbf{c}.$$

Every infinitely small rotational tensor may be expressed in this form.

If an arbitrary tensor $\mathbf{T}$ differs little from I, i.e. if $\mathbf{T}\cdot\mathbf{r}$ differs little from $\mathbf{r}$, then $\mathbf{T} - \mathrm{I}$ (where I stands for $\mathbf{a}^*\mathbf{a} + \mathbf{b}^*\mathbf{b} + \mathbf{c}^*\mathbf{c}$) is a small tensor. Let us resolve it into the sum of a small symmetric and a small asymmetric tensor

$$\mathbf{T} - \mathrm{I} = \mathbf{T}_1 + \mathbf{T}_2$$

or

$$\mathbf{T} = \mathrm{I} + \mathbf{T}_1 + \mathbf{T}_2.$$

Now $\mathbf{T}_1\cdot\mathbf{T}_2$ is a tensor of higher order than $\mathbf{T}_1$ and $\mathbf{T}_2$. If we can neglect it in comparison with $\mathbf{T}_1$ and $\mathbf{T}_2$, then we can also write

$$\begin{aligned}\mathbf{T} &= \mathrm{I} + \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_1\cdot\mathbf{T}_2\\ &= (\mathrm{I} + \mathbf{T}_1)(\mathrm{I} + \mathbf{T}_2),\end{aligned}$$

where now $\mathrm{I} + \mathbf{T}_1$ is the symmetric portion of the tensor $\mathbf{T}$, and neglecting quantities of higher order, $\mathbf{T}_2$ is a rotational tensor which arises from the asymmetric portion of $\mathbf{T}$ by the addition of I.

This effects the resolution of $\mathbf{T}$ into a product of a symmetrical and a rotational tensor in simple fashion from the resolution into a sum of a symmetric and an asymmetric tensor, which has already been shown to be possible in a more general manner.

Let x, y, z be the coefficients of a variable vector $\mathbf{r}$ referred to three mutually independent vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}.$$

From these coefficients by means of the three linear equations

$$\begin{aligned}\xi &= a_{11}x + a_{21}y + a_{31}z,\\ \eta &= a_{12}x + a_{22}y + a_{32}z,\\ \zeta &= a_{13}x + a_{23}y + a_{33}z,\end{aligned}$$

let three other quantities ξ, η, ζ be derived such that they lead to a second vector $\mathbf{r}'$ referred to the same vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$:

$$\mathbf{r}' = \xi\mathbf{a} + \eta\mathbf{b} + \zeta\mathbf{c}.$$

Inserting the linear expressions for ξ, η, ζ we get

$$\mathbf{r}' = x\mathbf{e} + y\mathbf{f} + z\mathbf{g}$$

where

$$\begin{aligned} \mathbf{e} &= a_{11}\mathbf{a} + a_{12}\mathbf{b} + a_{13}\mathbf{c} \\ \mathbf{f} &= a_{21}\mathbf{a} + a_{22}\mathbf{b} + a_{23}\mathbf{c} \\ \mathbf{g} &= a_{31}\mathbf{a} + a_{32}\mathbf{b} + a_{33}\mathbf{c}. \end{aligned}$$

Accordingly the passage from $\mathbf{r}$ to $\mathbf{r}'$ is effected by means of the tensor

$$\mathbf{T} = \mathbf{ea}^* + \mathbf{fb}^* + \mathbf{gc}^*,$$

which transforms the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ into $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$. Inserting the expressions for $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ the form of $\mathbf{T}$ becomes

$$\begin{aligned} \mathbf{T} = a_{11}\mathbf{aa}^* + a_{22}\mathbf{bb}^* + a_{33}\mathbf{cc}^* + a_{23}\mathbf{cb}^* + a_{32}\mathbf{bc}^* + a_{31}\mathbf{ac}^* \\ + a_{13}\mathbf{ca}^* + a_{12}\mathbf{ba}^* + a_{21}\mathbf{ab}^*. \end{aligned}$$

The nine coefficients $a_{\alpha\beta}$ refer to the nine tensors $\mathbf{aa}^*$, $\mathbf{bb}^*$, $\mathbf{cc}^*$, $\mathbf{cb}^*$, etc., from which $\mathbf{T}$ is numerically derived.

Every system of three arbitrary linear functions of three variables which are regarded as coefficients with reference to three mutually independent vectors therefore corresponds to a definite tensor. Every tensor may be obtained in this manner from any system of three mutually independent vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. It is merely necessary to throw it into the form

$$\mathbf{T} = \mathbf{ea}^* + \mathbf{fb}^* + \mathbf{gc}^*$$

which is always possible by expressing the second vector in each term through $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ and then arranging the terms according to the second factor. The coefficients of $\mathbf{e}$, $\mathbf{f}$, $\mathbf{g}$ in relation to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ provide the coefficients of the linear functions by interchanging the rows and columns.

§ 13. Reversals and Rotations

A reversal about the $\mathbf{i}$-axis (i.e. a rotation through two right angles) changes the vectors $\mathbf{j}$, $\mathbf{k}$ into $-\mathbf{j}$, $-\mathbf{k}$, while $\mathbf{i}$

remains unaltered. It is therefore represented by the tensor

$$\mathbf{ii} - \mathbf{jj} - \mathbf{kk},$$

which may also be written

$$2\mathbf{ii} - \mathbf{ii} - \mathbf{jj} - \mathbf{kk} \text{ or } 2\mathbf{ii} - 1.$$

We have written 1 for the tensor

$$\mathbf{T} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$$

since it transforms **i, j, k** back into **i, j, k** and therefore also $\mathbf{T}\,.\,\mathbf{r}$ becomes **r**. Two reversals about different axes combine together into one rotation. Let us combine the two reversal tensors

$$2\mathbf{ii} - 1 \text{ and } 2\mathbf{i}'\mathbf{i}' - 1$$

to give the rotational tensor

$$\begin{aligned} \mathbf{T} &= (2\mathbf{i}'\mathbf{i}' - 1)(2\mathbf{ii} - 1) \\ &= 4\mathbf{i}'(\mathbf{i}'\,.\,\mathbf{i})\mathbf{i} - 2\mathbf{ii} - 2\mathbf{i}'\mathbf{i}' + 1. \end{aligned}$$

If all the terms be regarded as external products, then **ii** and $\mathbf{i}'\mathbf{i}'$ and $\mathbf{ii} + \mathbf{jj} + \mathbf{kk}$ vanish and we get the vectorial area

$$4(\mathbf{i}'\,.\,\mathbf{i})\mathbf{i}'\mathbf{i}.$$

As we found above, this vectorial area is at right angles to the axis of rotation, the rotation taking place in the sense of the opposite vectorial area:

$$4(\mathbf{i}'\,.\,\mathbf{i})\mathbf{ii}',$$

and its numerical value is equal to twice the sine of the angle of rotation, where the rotation is conceived as occurring so that the angle of rotation is less than 180°. Denoting by a the angle between **i** and $\mathbf{i}'$, which may be assumed smaller than 90° since in place of $\mathbf{i}'$ we may put $-\mathbf{i}'$ without altering the tensor, then the numerical value of

$$4(\mathbf{i}'\,.\,\mathbf{i})\mathbf{ii}'$$

equals

$$4 \cos a \sin a = 2 \sin 2a,$$

so that the angle between **i** and $\mathbf{i}'$ equals half the angle of rotation. In this way every rotation may be replaced by two reversals. We have merely to select **i** and $\mathbf{i}'$ both per-

pendicular to the axis of rotation, both of length unity, the angle between them being half the angle of rotation (which may be assumed less than or equal to 180°) and the order so chosen that the sense of the vectorial area **ii′** corresponds to the direction of rotation.

Taking the mirror image in the **jk**-plane transforms **i** to − **i** while **j** and **k** remain unaffected, and therefore it represents the tensor:

$$-\mathbf{ii}+\mathbf{jj}+\mathbf{kk} \text{ or } -2\mathbf{ii}+\mathrm{I}.$$

This tensor is the exact opposite of that corresponding to the reversal $2\mathbf{ii}-\mathrm{I}$. In the rotational tensor

$$\mathbf{T}=(2\mathbf{i}'\mathbf{i}'-\mathrm{I})(2\mathbf{ii}-\mathrm{I})$$

we may change both factors into their opposites without altering the tensor:

$$\begin{aligned}\mathbf{T}&=(-2\mathbf{i}'\mathbf{i}'+\mathrm{I})(-2\mathbf{ii}+\mathrm{I})\\&=4\mathbf{i}'(\mathbf{i}'\cdot\mathbf{i})\mathbf{i}-2\mathbf{ii}-2\mathbf{i}'\mathbf{i}'+\mathrm{I}.\end{aligned}$$

Thus both reversals **i** and **i′** can be replaced by mirror imaging in the planes at right angles to **i** and **i′**.

Let us separate the rotational tensor into its symmetric and its asymmetric portions:

$$\begin{aligned}\mathbf{T}_1&=2(\mathbf{i}'\cdot\mathbf{i})(\mathbf{i}'\mathbf{i}+\mathbf{ii}')-2\mathbf{ii}-2\mathbf{i}'\mathbf{i}'+\mathrm{I}\\\mathbf{T}_2&=2(\mathbf{i}'\cdot\mathbf{i})(\mathbf{i}'\mathbf{i}-\mathbf{ii}').\end{aligned}$$

The rotation is completely given with the vectorial area **ii′**, its numerical value determining the sine of half the angle of rotation, its normal the axis of rotation, and its *sense* that of the rotation. We propose to make these facts apparent in the expression for $\mathbf{T}_1$ and $\mathbf{T}_2$, by setting the *representation* of the vectorial area **ii′** equal to $\mathbf{i}\times\mathbf{i}'=\sin\vartheta/2\mathbf{k}$ and introducing **i**, **j**, **k**. For the sake of brevity let us write:

$$\mathbf{i}\cdot\mathbf{i}'=\cos\vartheta/2=\rho\,;\ \sin\vartheta/2=\sigma.$$

Then:

$$\mathbf{i}'=\rho\mathbf{i}+\sigma\mathbf{j}.$$

Introducing this expression for **i′** into $\mathbf{T}_1$ we get at once:

$$\begin{aligned}2\rho(\mathbf{i}'\mathbf{i}+\mathbf{ii}')&=4\rho^2\mathbf{ii}+2\rho\sigma(\mathbf{ji}+\mathbf{ij})\\2\mathbf{i}'\mathbf{i}&=2\rho^2\mathbf{ii}+2\rho\sigma(\mathbf{ji}+\mathbf{ij})+2\sigma^2\mathbf{jj}\\\mathbf{i}'\mathbf{i}-\mathbf{ii}'&=\sigma(\mathbf{ji}-\mathbf{ij}).\end{aligned}$$

Consequently

$$\begin{aligned} \mathbf{T}_1 &= -2\sigma^2(\mathbf{ii}+\mathbf{jj}) + 1 = 1 - 2\sigma^2 + 2\sigma^2\mathbf{kk}, \\ \mathbf{T}_2 &= 2\rho\sigma(\mathbf{ji}-\mathbf{ij}), \end{aligned}$$

the result we have been seeking. In the expression for $\mathbf{T}_1$ there merely occurs the vector $\mathbf{i}\times\mathbf{i}' = \sigma\mathbf{k}$ and the numerical quantity σ. The transformation $\mathbf{T}_1 \cdot \mathbf{r}$ consists of

$$(1-2\sigma^2)\mathbf{r} + 2\sigma^2\mathbf{k}(\mathbf{k}\cdot\mathbf{r}).$$

Hence if $\mathbf{r}$ be supposed drawn out from O, its end-point P is drawn towards the axis of rotation in the ratio:

$$1 - 2\sigma^2 = \rho^2 - \sigma^2 = \cos\vartheta.$$

This is immediately evident from fig. 33. If, in fact, we make $OQ/OP = \cos\vartheta$ then:

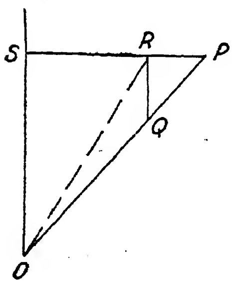

Fig. 33.

$$OQ = (1-2\sigma^2)\mathbf{r}; \quad QP = 2\sigma^2\mathbf{r};$$
$$OS = \mathbf{k}(\mathbf{k}\cdot\mathbf{r}); \quad QR/OS = QP/OP;$$

so that: $QR = 2\sigma^2\mathbf{k}(\mathbf{k}\cdot\mathbf{r})$,

and consequently:

$$\mathbf{T}_1 \cdot \mathbf{r} = OQ + QR = OR.$$

On the other hand:

$$\begin{aligned} \mathbf{T}_2 \cdot \mathbf{r} &= 2\rho\sigma(\mathbf{ji}-\mathbf{ij})\cdot\mathbf{r} \\ &= 2\rho\sigma\mathbf{r}\times\mathbf{k} \end{aligned}$$

is merely the vectorial product of $\mathbf{r}$ by $2\rho\mathbf{i}\times\mathbf{i}'$. In this way $\mathbf{T}_1$ and $\mathbf{T}_2$ are both expressed by the number $\rho = \cos\vartheta/2$ and by the vector $\mathbf{i}\times\mathbf{i}'$:

$$\begin{aligned} \mathbf{T}_1 &= 2\rho^2 - 1 + 2(\mathbf{i}\times\mathbf{i}')(\mathbf{i}\times\mathbf{i}'), \\ \mathbf{T}_2\cdot\mathbf{r} &= 2\rho\mathbf{r}\times(\mathbf{i}\times\mathbf{i}'). \end{aligned}$$

If $\mathbf{T}$ be referred to an arbitrary system $\bar{\mathbf{i}}, \bar{\mathbf{j}}, \bar{\mathbf{k}}$ then there occurs in $\mathbf{T}$, in addition to ρ, the three coefficients of the vectorial area $\mathbf{ii}'$ or its *representation:*

$$\mathbf{i}\times\mathbf{i}' = \lambda\bar{\mathbf{i}} + \mu\bar{\mathbf{j}} + \nu\bar{\mathbf{k}}.$$

Since the numerical value equals σ we have:

$$\rho^2 + \lambda^2 + \mu^2 + \nu^2 = 1.$$

Accordingly

$$\begin{aligned}\mathbf{T}_1 = &(2\rho^2 - 1)(\bar{\mathbf{i}}\mathbf{i} + \bar{\mathbf{j}}\mathbf{j} + \bar{\mathbf{k}}\mathbf{k}) \\ &+ 2(\lambda^2\bar{\mathbf{i}}\bar{\mathbf{i}} + \mu^2\bar{\mathbf{j}}\bar{\mathbf{j}} + \nu^2\bar{\mathbf{k}}\bar{\mathbf{k}}) \\ &+ 2\mu\nu(\bar{\mathbf{j}}\mathbf{k} + \bar{\mathbf{k}}\mathbf{j}) + 2\nu\lambda(\bar{\mathbf{k}}\mathbf{i} + \bar{\mathbf{i}}\mathbf{k}) + 2\lambda\mu(\bar{\mathbf{i}}\mathbf{j} + \bar{\mathbf{j}}\mathbf{i}),\end{aligned}$$

and

$$\mathbf{T}_2 = -2\rho[\lambda(\bar{\mathbf{j}}\mathbf{k} - \bar{\mathbf{k}}\mathbf{j}) + \mu(\bar{\mathbf{k}}\mathbf{i} - \bar{\mathbf{i}}\mathbf{k}) + \nu(\bar{\mathbf{i}}\mathbf{j} - \bar{\mathbf{j}}\mathbf{i})].$$

The system of nine coefficients then become equal to:

$$\begin{array}{lll} (2\rho^2 - 1) + 2\lambda^2, & 2\lambda\mu + 2\rho\nu, & 2\nu\lambda - 2\rho\mu, \\ 2\lambda\mu - 2\rho\nu, & 2\rho^2 - 1 + 2\mu^2, & 2\mu\nu + 2\rho\lambda, \\ 2\nu\lambda + 2\rho\mu, & 2\mu\nu - 2\rho\lambda, & 2\rho^2 - 1 + 2\nu^2. \end{array}$$

These nine quantities are the coefficients of the three vectors into which $\bar{\mathbf{i}}, \bar{\mathbf{j}}, \bar{\mathbf{k}}$ transform during the rotation. They give the formulæ for transforming from one system of rectangular coordinate into another, to use the language of analytical geometry, and were given in this form by Euler.

From the standpoint of vector analysis they are embodied in the expression

$$\mathbf{T} = (2\rho^2 - 1) + 2(\mathbf{i} \times \mathbf{i}')(\mathbf{i} \times \mathbf{i}') + 2\rho(\mathbf{i}'\mathbf{i} - \mathbf{i}\mathbf{i}')$$

which can be referred to an arbitrary system of vectors.

In an affine transformation let the vectors **a, b, c** be transformed into **e, f, g** so that it is represented by the tensor:

$$\mathbf{T} = \mathbf{e}\mathbf{a}^* + \mathbf{f}\mathbf{b}^* + \mathbf{g}\mathbf{c}^*.$$

The vectorial areas **bc, ca, ab** are then transformed into the vectorial areas **fg, ge, ef,** but it does not follow that the *representations* of the vectorial areas transform into the *representations* of **fg, ge, ef.** For a vector standing at right angles to a vectorial area will in general not be at right angles after transformation. The relation between the representation of a vectorial area **P** and of a vectorial area **P** into which the former transforms will not be provided by the tensor

$$\mathbf{T} = \mathbf{e}\mathbf{a}^* + \mathbf{f}\mathbf{b}^* + \mathbf{g}\mathbf{c}^*,$$

but by another tensor which transforms the vectors

$$\mathbf{b} \times \mathbf{c}, \quad \mathbf{c} \times \mathbf{a}, \quad \mathbf{a} \times \mathbf{b}$$

into the vectors:

$$\mathbf{f} \times \mathbf{g}, \quad \mathbf{g} \times \mathbf{e}, \quad \mathbf{e} \times \mathbf{f}.$$

Now $$\mathbf{b} \times \mathbf{c} = \mathbf{a}^*(\mathbf{abc}), \text{ etc.}$$

Hence we may write this tensor:

$$\frac{\mathbf{efg}}{\mathbf{abc}}(\mathbf{e}^*\mathbf{a} + \mathbf{f}^*\mathbf{b} + \mathbf{g}^*\mathbf{c}).$$

The expression in the brackets is conjugate to the tensor

$$\mathbf{T}^{-1} = \mathbf{ae}^* + \mathbf{bf}^* + \mathbf{cg}^*,$$

which transforms the vectors **e, f, g** into **a, b, c** and is therefore reciprocal to **T**. Accordingly we may write:

$$\overline{\mathbf{T}}^{-1} = \mathbf{e}^*\mathbf{a} + \mathbf{f}^*\mathbf{b} + \mathbf{g}^*\mathbf{c}.$$

The tensor which changes the representation of a vectorial area into the representation of the transformed vectorial area for an affine transformation **T**, is therefore equal to the ratio $\frac{\mathbf{efg}}{\mathbf{abc}}$ by which a volume is altered, multiplied by the tensor $\overline{\mathbf{T}}^{-1}$, i.e. the tensor which is reciprocal to the conjugate of **T**, or, as we may also express it, which is conjugate to the reciprocal of **T**. For, the tensor conjugate to **T**

$$\mathbf{a}^*\mathbf{e} + \mathbf{b}^*\mathbf{f} + \mathbf{c}^*\mathbf{g}$$

changes $\mathbf{e}^*$, $\mathbf{f}^*$, $\mathbf{g}^*$ into $\mathbf{a}^*$, $\mathbf{b}^*$ $\mathbf{c}^*$, and therefore its reciprocal changes $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ into $\mathbf{e}^*$, $\mathbf{f}^*$, $\mathbf{g}^*$, and therefore has the form

$$\mathbf{e}^*\mathbf{a} + \mathbf{f}^*\mathbf{b} + \mathbf{g}^*\mathbf{c}.$$

For a symmetric tensor $\mathbf{T} = \overline{\mathbf{T}}$, and hence $\overline{\mathbf{T}}^{-1} = \mathbf{T}^{-1}$; here then the reciprocal tensor changes the representation of a vectorial area into the representation of the transformed vectorial area, if for the moment we ignore the numerical factor $\frac{\mathbf{efg}}{\mathbf{abc}}$ which would merely indicate a geometrical magnification or diminution and into which an inversion would also enter if $\frac{\mathbf{efg}}{\mathbf{abc}}$ were negative, i.e. if the

sequence of the three vectors of the system is altered by the transformation.

Consider, for example, a sphere which is transformed by a symmetric tensor into an ellipsoid of like volume without altering its *sense*. An element of the spherical surface transforms into an element of that of the ellipsoid, but the normals of the spherical elements do not transform into the normals of the ellipsoid but into the corresponding diameters. On the other hand, the reciprocal tensor changes the normal of the sphere into the normal of the ellipsoid, for the point in question and the ratio of the lengths is the same as that of the surface elements.

For a rotation, or one, **T**, associated with an inversion, as we have already seen, the conjugate tensor $\bar{\mathbf{T}}$ is identical with the reciprocal, and therefore

$$\bar{\mathbf{T}}^{-1} = \mathbf{T}$$

and at the same time

$$\mathbf{efg} = \pm\,\mathbf{abc}.$$

According as an inversion is or is not also associated with the rotation, the representation of a vectorial area will be opposite or equal to the representation of the transformed vectorial area. One recognises that this case can arise only with these tensors. For, from $\bar{\mathbf{T}}^{-1} = \mathbf{T}$ it follows that $\bar{\mathbf{T}} = \mathbf{T}^{-1}$, and this can occur with no other tensor.

§ 14. Tensor Fields

Just as we have passed from the conception of a vector to the conception of a vector field, so we may proceed to the idea of a tensor field. At each point of space or element of space imagine a tensor given; it may be presumed to vary from point to point. Corresponding to a system of elastic bodies under stress, for example, there is a tensor field of symmetric tensors which represents the state of stress at each point. If every portion were affected in exactly the same manner, the tensor would be everywhere the same and the tensor field would be constant. In

general, however, it varies from one position to another. Imagine the symmetric tensor which belongs to any point, thrown into the form

$$\mathbf{T} = \lambda_1\mathbf{ii} + \lambda_2\mathbf{jj} + \lambda_3\mathbf{kk},$$

then $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the directions of principal stress. On a vectorial area

$$d\mathfrak{a}_1 = d\omega\mathbf{jk}$$

of size $d\omega$ the force at right angles to it is

$$\mathbf{T}d\mathfrak{a}_1 = \mathbf{T} \cdot d\omega\mathbf{i} = \lambda_1 d\omega\mathbf{i}.$$

A positive value of λ would indicate a force which operates in the direction of the representation of $d\mathfrak{a} = d\omega\mathbf{jk}$. If then we make the assumption that on making a section in the body the sense of the vectorial area will be such that the portion of the body considered lies on its positive side, then a positive value of λ_1 corresponds to a pressure and a negative value to a tension. In the same way the vectorial area $d\omega\mathbf{ki}$ experiences a force

$$\mathbf{T}d\mathfrak{a}_2 = \mathbf{T} \cdot d\omega\mathbf{j} = \lambda_2 d\omega\mathbf{j},$$

and the vectorial area $d\omega\mathbf{ij}$ a force

$$\mathbf{T}d\mathfrak{a}_3 = \mathbf{T} \cdot d\omega\mathbf{k} = \lambda_3 d\omega\mathbf{k}.$$

On each of these vectorial areas the force in each instance acts at right angles. For any other indefinitely small vectorial area $d\mathfrak{a}$, the force acting on it is $\mathbf{T}d\mathfrak{a}$. For the vectorial area $d\mathfrak{a}$ may be conceived as one of the four surfaces forming the tetrahedron the other sides of which, $d\mathfrak{a}_1$, $d\mathfrak{a}_2$, $d\mathfrak{a}_3$, are parallel to $\mathbf{jk}$, $\mathbf{ki}$, $\mathbf{ij}$. If on all the surfaces the *sense* is taken in such a way that the tetrahedron lies on the positive side, then

$$d\mathfrak{a} + d\mathfrak{a}_1 + d\mathfrak{a}_2 + d\mathfrak{a}_3 = 0$$

and therefore

$$\mathbf{T}d\mathfrak{a} + \mathbf{T}d\mathfrak{a}_1 + \mathbf{T}d\mathfrak{a}_2 + \mathbf{T}d\mathfrak{a}_3 = 0.$$

The last three terms are the forces acting on the corresponding sides. Since the force acting on $d\mathfrak{a}$ is in equilibrium with these it must be $\mathbf{T}d\mathfrak{a}$.

In the same way the deformation at each point determines a tensor field. Consider, for example, a point O of the body in the undeformed state and imagine three mutually independent but very small vectors $\epsilon\mathbf{a}$, $\epsilon\mathbf{b}$, $\epsilon\mathbf{c}$ drawn out from O, then the material particles of the body lying on these lines will, after the deformation, lie on another set of three small vectors $\epsilon\mathbf{e}$, $\epsilon\mathbf{f}$, $\epsilon\mathbf{g}$. The whole deformation in the neighbourhood of O is, to a first approximation, affine and is therefore represented by the tensor

$$\mathbf{T} = \mathbf{ea}^* + \mathbf{fb}^* + \mathbf{gc}^*$$

which changes the vectors $\epsilon\mathbf{a}$, $\epsilon\mathbf{b}$, $\epsilon\mathbf{c}$ into $\epsilon\mathbf{e}$, $\epsilon\mathbf{f}$, $\epsilon\mathbf{g}$. The smallness of the vectors we here attribute to ϵ so that the tensor is composed of finite vectors.

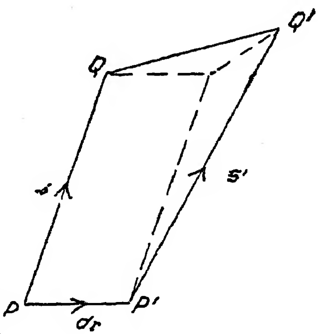

FIG. 34.

Let the deformation of a body be specified by a vector **s** at each of its original points, the vector representing the displacement of the point into its new position. Suppose **s** given as a function of the position vector **r** drawn from a fixed point O to the point under consideration, then a small change $d\mathbf{s}$ will correspond to a small change $d\mathbf{r}$. However the form of the relation between **s** and **r**, the relation between the indefinitely small vectors will be expressed by linear equations between their coefficients, and therefore the relation between $d\mathbf{s}$ and $d\mathbf{r}$ will be expressed by a tensor **T**, so that:

$$d\mathbf{s} = \mathbf{T} \cdot d\mathbf{r}.$$

We might regard **T** as analogous to a differential coefficient except that we are dealing here with the relation between two vectors which are differentiated.

If now the displacement vector **s** joins P to Q, if,

moreover, $d\mathbf{r}$ joins P to P′ and $\mathbf{s}' = \mathbf{s} + d\mathbf{s}$ joins P′ to Q′ (fig. 34), then the vector QQ′ equals

$$d\mathbf{r} + d\mathbf{s} = (1 + \mathbf{T}) \cdot d\mathbf{r}.$$

The tensor $1 + \mathbf{T}$ thus expresses the relation between the undeformed vector PP′ and the deformed vector QQ′. If for numerical calculation we desire this tensor $1 + \mathbf{T}$ referred to a system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we must in the first instance suppose $\mathbf{r}$ and $\mathbf{s}$ derived numerically from $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
$$\mathbf{s} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k},$$

and u, v, w written down as functions of x, y, z. The three vectors

$$dx\mathbf{i}\,;\ dy\mathbf{j}\,;\ dz\mathbf{k}$$

are changed by the tensor $\mathbf{T}$ into

$$\frac{\partial \mathbf{s}}{\partial x} dx\,;\ \frac{\partial \mathbf{s}}{\partial y} dy\,;\ \frac{\partial \mathbf{s}}{\partial z} dz.$$

Hence $$\mathbf{T} = \frac{\partial \mathbf{s}}{\partial x}\mathbf{i} + \frac{\partial \mathbf{s}}{\partial y}\mathbf{j} + \frac{\partial \mathbf{s}}{\partial z}\mathbf{k}$$

and $$1 + \mathbf{T} = \left(\mathbf{i} + \frac{\partial \mathbf{s}}{\partial x}\right)\mathbf{i} + \left(\mathbf{j} + \frac{\partial \mathbf{s}}{\partial y}\right)\mathbf{j} + \left(\mathbf{k} + \frac{\partial \mathbf{s}}{\partial z}\right)\mathbf{k}.$$

The nine coefficients of the tensor $1 + \mathbf{T}$ referred to i, j, k are accordingly

$$\begin{array}{ccc} 1 + \dfrac{\partial u}{\partial x}, & \dfrac{\partial v}{\partial x}, & \dfrac{\partial w}{\partial x}, \\[2ex] \dfrac{\partial u}{\partial y}, & 1 + \dfrac{\partial v}{\partial y}, & \dfrac{\partial w}{\partial y}, \\[2ex] \dfrac{\partial u}{\partial z}, & \dfrac{\partial v}{\partial z}, & 1 + \dfrac{\partial w}{\partial z}. \end{array}$$

Resolving $1 + \mathbf{T}$ into a symmetric and an asymmetric portion

$$1 + \mathbf{T} = 1 + \mathbf{T}_1 + \mathbf{T}_2,$$

where

$$\mathbf{T}_1 = \frac{\mathbf{T} + \overline{\mathbf{T}}}{2}; \ \mathbf{T}_2 = \frac{\mathbf{T} - \overline{\mathbf{T}}}{2}.$$

If $\mathbf{T}_1$ and $\mathbf{T}_2$ are so small that vectors of the second order in comparison with $\mathbf{T}_1 \cdot \mathbf{r}$ and $\mathbf{T}_2 \cdot \mathbf{r}$ may be neglected, then, as we have remarked above, we may put

$$\begin{aligned} 1 + \mathbf{T} &= (1 + \mathbf{T}_1)(1 + \mathbf{T}_2) \\ &= (1 + \mathbf{T}_2)(1 + \mathbf{T}_1) \end{aligned}$$

and we may regard $1 + \mathbf{T}_2$ as a rotational tensor. Since a rotation does not affect the distance between two points, and therefore does not deform the body, we may term

$$1 + \mathbf{T}_1$$

the deformation tensor. As indicated above, $\mathbf{T}_1$ may be thrown into the form

$$\lambda_1 \mathbf{i}'\mathbf{i}' + \lambda_2 \mathbf{j}'\mathbf{j}' + \lambda_3 \mathbf{k}'\mathbf{k}',$$

then $1 + \mathbf{T}_1$ assumes the form

$$(1 + \lambda_1)\mathbf{i}'\mathbf{i}' + (1 + \lambda_2)\mathbf{j}'\mathbf{j}' + (1 + \lambda_3)\mathbf{k}'\mathbf{k}',$$

i.e. the deformation of the element of the body consists of three extensions $(\lambda > 0)$ or compressions $(\lambda < 0)$ along the three axis determined by $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$.

The whole change in position of the element in the neighbourhood of P, the end point of $\mathbf{r}$, consists of a parallel displacement of P to Q, a rotation $1 + \mathbf{T}_2$ and extensions or compressions $1 + \mathbf{T}_1$. The sum of the three extensions or compressions

$$1 + \lambda_1 + 1 + \lambda_2 + 1 + \lambda_3$$

equals the value derived from each form of the tensor $1 + \mathbf{T}$ if the two vectors in each term are multiplied scalarly together, that is to say, it equals

$$\left(\mathbf{i} + \frac{\partial \mathbf{s}}{\partial x}\right) \cdot \mathbf{i} + \left(\mathbf{j} + \frac{\partial \mathbf{s}}{\partial y}\right) \cdot \mathbf{j} + \left(\mathbf{k} + \frac{\partial \mathbf{s}}{\partial z}\right) \cdot \mathbf{k},$$

i.e.

$$1 + \frac{\partial u}{\partial x} + 1 + \frac{\partial v}{\partial y} + 1 + \frac{\partial w}{\partial z},$$

or

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{s}.$$

The ratio for the space transformation for the tensor $1 + \mathbf{T}$ is equal to the external product:

$$\left(\mathbf{i} + \frac{\partial \mathbf{s}}{\partial x}\right)\left(\mathbf{j} + \frac{\partial \mathbf{s}}{\partial y}\right)\left(\mathbf{k} + \frac{\partial \mathbf{s}}{\partial z}\right).$$

Neglecting terms of the second order in the expansion of this expression, we get

$$\mathbf{ijk} + \frac{\partial \mathbf{s}}{\partial x}\mathbf{jk} + \mathbf{i}\frac{\partial \mathbf{s}}{\partial y}\mathbf{k} + \mathbf{ij}\frac{\partial \mathbf{s}}{\partial z},$$

i.e.

$$1 + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 1 + (\lambda_1 + \lambda_2 + \lambda_3),$$

so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot s$$

is the fraction by which the volume element at P is increased or diminished according as $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ is positive or negative.

If, for example, we consider the case where

$$\mathbf{s} = \epsilon\mathbf{i}(\mathbf{k} \cdot \mathbf{r}),$$

i.e. where all the points in the **ij**-plane ($\mathbf{k} \cdot \mathbf{r} = 0$) are not displaced, while every point P outside the plane is displaced parallel to **i**, by an amount $\epsilon(\mathbf{k} \cdot \mathbf{r}) = \epsilon z$ which is proportional to the distance z from the **ij**-plane but of opposite signs on opposite sides of the plane (fig. 35), then

$$d\mathbf{s} = \epsilon\mathbf{i}(\mathbf{k} \cdot d\mathbf{r})$$

and

$$\mathrm{I} + \mathbf{T} = \mathrm{I} + \epsilon\mathbf{ik},$$

$$\mathbf{T}_1 = \frac{\epsilon}{2}(\mathbf{ik} + \mathbf{ki})$$

$$\mathbf{T}_2 = \frac{\epsilon}{2}(\mathbf{ik} - \mathbf{ki}).$$

A cube, one of whose corners lies at O and whose edges proceeding from O coincide with the vectors **i**, **j**, **k**, will be transformed into a parallelepiped. The corner O and the edges from O parallel to **i** and **j** remain unaltered. The edge parallel to **k** is changed into

$$(\mathrm{I} + \mathbf{T})\mathbf{k} = \mathbf{k} + \epsilon\mathbf{i}.$$

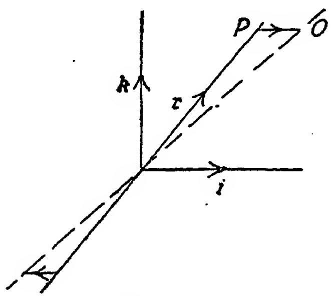

Fig. 35.

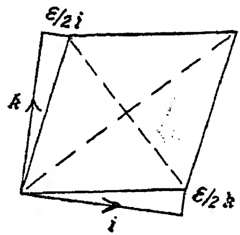

Fig. 36.

This is termed a *shearing* of the body. This alteration of the cube, if ϵ is sufficiently small to allow a neglect of terms of the second order in ϵ, may be supposed constituted, on the one hand, of a rotation $\mathrm{I} + \mathbf{T}_2$ about the axis of **j** in the *sense* of **ki** through an angle $\epsilon/2$ and on the other hand of a deformation

$$\mathrm{I} + \mathbf{T}_1 = \mathrm{I} + \frac{\epsilon}{2}(\mathbf{ik} + \mathbf{ki})$$

$$= (\mathbf{i} + \frac{\epsilon}{2}\mathbf{k})\mathbf{i} + \mathbf{jj} + (\mathbf{k} + \frac{\epsilon}{2}\mathbf{i})\mathbf{k},$$

which transforms **i** and **k** into $\mathbf{i} + \frac{\epsilon}{2}\mathbf{k}$ and $\mathbf{k} + \frac{\epsilon}{2}\mathbf{i}$ while **j**

remains unaltered. An alternative form for this deformation tensor is

$$(1 + \epsilon/2)\frac{(\mathbf{i} + \mathbf{k})(\mathbf{i} + \mathbf{k})}{2} + (1 - \epsilon/2)\frac{(\mathbf{k} - \mathbf{i})(\mathbf{k} - \mathbf{i})}{2} + \mathbf{jj},$$

i.e. it transforms the self-reciprocal system

$$\frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}, \mathbf{j}, \frac{\mathbf{k} - \mathbf{i}}{\sqrt{2}}$$

into $$(1 + \epsilon/2)\frac{\mathbf{i} + \mathbf{k}}{\sqrt{2}}, \mathbf{j}, (1 - \epsilon/2)\frac{\mathbf{k} - \mathbf{i}}{\sqrt{2}};$$

or it consists of an extension $1 + \epsilon/2$ parallel to $\mathbf{i} + \mathbf{k}$ and a compression $1 - \epsilon/2$ parallel to $\mathbf{k} - \mathbf{i}$ (fig. 36). The ratio of the corresponding volumes is

$$\frac{(\mathbf{i} + \epsilon/2\mathbf{k})\mathbf{j}(\mathbf{k} + \epsilon/2\mathbf{i})}{\mathbf{ijk}} = 1 - \epsilon^2/4,$$

that is to say, if terms of the second order be neglected the volume is unaltered by the deformation. The change is merely one of shape and not of volume.

Consider, on the other hand, the displacement vector $\mathbf{s}$:

$$\mathbf{s} = \epsilon\mathbf{k}(\mathbf{k} \cdot \mathbf{r}).$$

Once more all points in the $\mathbf{ij}$-plane remain unaffected ($\mathbf{k} \cdot \mathbf{r} = 0$); every point P outside the plane is displaced parallel to $\mathbf{k}$, that is perpendicular to the plane by an amount

$$\epsilon(\mathbf{k} \cdot \mathbf{r}) = \epsilon z,$$

that is, proportional to the distance from the plane but in opposite directions on opposite sides. Then

$$d\mathbf{s} = \epsilon\mathbf{k}(\mathbf{k} \cdot d\mathbf{r}),$$

and

$$1 + \mathrm{T} = 1 + \epsilon\mathbf{kk} = \mathbf{ii} + \mathbf{jj} + (1 + \epsilon)\mathbf{kk}.$$

This is a symmetric tensor which changes $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into $\mathbf{i}$, $\mathbf{j}$, $(1 + \epsilon)\mathbf{k}$. The ratio of corresponding volumes is $1 + \epsilon$.

If now we turn to the general tensor

$$1 + \mathbf{T} = 1 + \frac{\partial \mathbf{s}}{\partial x}\mathbf{i} + \frac{\partial \mathbf{s}}{\partial y}\mathbf{j} + \frac{\partial \mathbf{s}}{\partial z}\mathbf{k}$$

and its symmetrical portion

$$1 + \mathbf{T}_1 = 1 + \frac{\mathbf{T} + \bar{\mathbf{T}}}{2},$$

the latter may be written

$$1 + \mathbf{T}_1 = \left(1 + \frac{\partial u}{\partial x}\right)\mathbf{ii} + \left(1 + \frac{\partial v}{\partial y}\right)\mathbf{jj} + \left(1 + \frac{\partial w}{\partial z}\right)\mathbf{kk}$$
$$+ \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)(\mathbf{jk} + \mathbf{kj}) + \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)(\mathbf{ki} + \mathbf{ik})$$
$$+ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)(\mathbf{ij} + \mathbf{ji}).$$

If now $\frac{\partial \mathbf{s}}{\partial x}, \frac{\partial \mathbf{s}}{\partial y}, \frac{\partial \mathbf{s}}{\partial z}$ are so small that the squares of their coefficients are negligible, then $1 + \mathbf{T}_1$ may be resolved into a product of six tensors:

$$1 + \frac{\partial u}{\partial x}\mathbf{ii},\ 1 + \frac{\partial v}{\partial y}\mathbf{jj},\ 1 + \frac{\partial w}{\partial z}\mathbf{kk},$$
$$1 + \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)(\mathbf{jk} + \mathbf{kj}),\ 1 + \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)(\mathbf{ki} + \mathbf{ik}),$$
$$1 + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)(\mathbf{ij} + \mathbf{ji}),$$

which may be applied in any order; for on multiplying out, the product becomes $1 + \mathbf{T}_1$ if terms up to the first order only are retained. Three of these are of the type

$$1 + \epsilon\mathbf{kk},$$

and three of the type

$$1 + \frac{\epsilon}{2}(\mathbf{ki} + \mathbf{ik}).$$

With an elastic body there exist relations between the six coefficients of the deformation tensor and the six coefficients of the stress tensor, by means of which the one set may be calculated from the other. For sufficiently small deformations the relations may be regarded as linear. In particular they become simplified when the body is homogeneous and isotropic, for in that event the deformation tensor $\mathbf{T}_1$ referred to a system **i, j, k** for which the stress tensor is thrown into the form

$$\lambda_1\mathbf{ii} + \lambda_2\mathbf{jj} + \lambda_3\mathbf{kk},$$

must likewise assume the same form

$$\mu_1\mathbf{ii} + \mu_2\mathbf{jj} + \mu_3\mathbf{kk}$$

and the operation of μ_2 and μ_3 on λ_1 will on grounds of symmetry be the same, and equal to that of μ_3 and μ_1 on λ_2, and μ_1 and μ_2 on λ_3.

From the equation

$$\lambda_1 = a\mu_1 + b(\mu_2 + \mu_3)$$

the remaining two may be derived by cyclic substitution of the indices, since the elastic properties are the same in the different directions. These equations may likewise be written in the form

$$\lambda_\alpha = (a - b)\mu_\alpha + b(\mu_1 + \mu_2 + \mu_3).$$

Denoting the stress tensor by **S** then:

$$\mathbf{S} = (a - b)\mathbf{T}_1 + b\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)(\mathbf{ii} + \mathbf{jj} + \mathbf{kk}).$$

In this **i, j, k** may be any arbitrary self-reciprocal system, and u, v, w, x, y, z the corresponding coefficients of **s** and **r**. For in each of these systems $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ has the same value. If $\mathbf{T}_1$ is also referred to this system, **i, j, k,** we derive the six coefficients of the tensor **S** expressed in terms of the six coefficients of $\mathbf{T}_1$. If we write $\nabla \cdot \mathbf{s}$ for

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

then the six coefficients of $\mathbf{S}$ assume the form

$$(a-b)\frac{\partial u}{\partial x}+b\nabla\cdot\mathbf{s},\ (a-b)\frac{\partial v}{\partial y}+b\nabla\cdot\mathbf{s},\ (a-b)\frac{\partial w}{\partial z}+b\nabla\cdot\mathbf{s}$$

$$\frac{a-b}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\ \frac{a-b}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\ \frac{a-b}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

and the stress on a vectorial area $d\mathbf{F}$ passing through any considered point, reckoned per unit of area is

$$\mathbf{S}\cdot\mathbf{n}=(a-b)\mathbf{T}_1\cdot\mathbf{n}+b(\nabla\cdot\mathbf{s})\mathbf{n},$$

where $d\omega\mathbf{n}$ is the *representation* of the vectorial area and $d\omega$ its numerical value.

The stress $\mathbf{S}\cdot\mathbf{n}$ may be described in the following manner for different directions of the normal $\mathbf{n}$ of unit length:

It consists of two components of which the one is the vector into which $(a-b)\mathbf{n}$ is transformed by the transformation $\mathbf{T}_1$, while the other is proportional to $\mathbf{n}$.

Let the flow of a fluid be given by the vector field of its velocity $\mathbf{v}$ as a function of the position vector $\mathbf{r}$ of each point. In the small element of time dt, the displacement vector which was represented by $\mathbf{s}$ when considering the deformation of an elastic body, equals

$$\mathbf{s}=\mathbf{v}dt.$$

Hence the vector $d\mathbf{r}$ during the element of time dt changes into the vector

$$d\mathbf{r}+d\mathbf{s}=d\mathbf{r}+d\mathbf{v}dt,$$

where $d\mathbf{v}$ is the change corresponding to $d\mathbf{r}$ when the time is constant.

Let the passage from the vector $d\mathbf{r}$ to $d\mathbf{v}$ be effected by the tensor $\mathbf{T}$,

$$d\mathbf{v}=\mathbf{T}\cdot d\mathbf{r},$$

then

$$d\mathbf{r}+d\mathbf{s}=(1+\mathbf{T}dt)\cdot d\mathbf{r}.$$

Let the tensor $\mathbf{T}$ be resolved into a symmetrical portion $\mathbf{T}_1$ and an asymmetrical portion $\mathbf{T}_2$, and write

$$1+\mathbf{T}_1dt+\mathbf{T}_2dt$$

in the form of a product

$$(\mathrm{I} + \mathbf{T}_1 dt)(\mathrm{I} + \mathbf{T}_2 dt),$$

or also

$$(\mathrm{I} + \mathbf{T}_2 dt)(\mathrm{I} + \mathbf{T}_1 dt).$$

The tensor $\mathrm{I} + \mathbf{T}_1 dt$ represents a pure deformation without rotation, while $\mathrm{I} + \mathbf{T}_2 dt$ is an indefinitely small rotation. If for purposes of calculation both tensors are referred to a system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ we must set:

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k},$$
$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If now

$$d\mathbf{v} = \mathbf{T} \cdot d\mathbf{r},$$

then $\mathbf{T}$ changes the vectors $dx\mathbf{i}$, $dy\mathbf{j}$, $dz\mathbf{k}$ into

$$\frac{\partial \mathbf{v}}{\partial x}dx,\ \frac{\partial \mathbf{v}}{\partial y}dy,\ \frac{\partial \mathbf{v}}{\partial z}dz,$$

and therefore

$$\mathbf{T} = \frac{\partial \mathbf{v}}{\partial x}\mathbf{i} + \frac{\partial \mathbf{v}}{\partial y}\mathbf{j} + \frac{\partial \mathbf{v}}{\partial z}\mathbf{k}.$$

Consequently

$$\mathbf{T}_1 = \frac{\partial u}{\partial x}\mathbf{ii} + \frac{\partial v}{\partial y}\mathbf{jj} + \frac{\partial w}{\partial z}\mathbf{kk} + \frac{\mathrm{I}}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)(\mathbf{jk} + \mathbf{kj})$$
$$+ \frac{\mathrm{I}}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)(\mathbf{ki} + \mathbf{ik}) + \frac{\mathrm{I}}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)(\mathbf{ij} + \mathbf{ji}),$$

$$\mathbf{T}_2 = \frac{\mathrm{I}}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)(\mathbf{kj} - \mathbf{jk}) + \frac{\mathrm{I}}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)(\mathbf{ik} - \mathbf{ki})$$
$$+ \frac{\mathrm{I}}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)(\mathbf{ji} - \mathbf{ij}).$$

The rotational tensor

$$\mathrm{I} + \mathbf{T}_2 dt,$$

as we have seen above, turns the element of fluid about an axis which is at right angles to the vectorial area:

$$\frac{\mathrm{I}}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)dt\mathbf{jk} + \frac{\mathrm{I}}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)dt\mathbf{ki} + \frac{\mathrm{I}}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dt\mathbf{ij}$$
$$= \tfrac{1}{2}\nabla\mathbf{v}\,dt,$$

by an angular amount equal to $d\vartheta$ which equals the numerical value of the vectorial area and in the direction corresponding to its sense. Dividing by dt we obtain in place of $d\vartheta$, the rotational velocity $\frac{d\vartheta}{dt}$. The vectorial area

$$\tfrac{1}{2}\nabla \mathbf{v},$$

or its representation

$$\tfrac{1}{2}\nabla \times \mathbf{v}$$

indicates, then, the axis of rotation, the direction of rotation and the rotational speed.

The vector $\nabla \times \mathbf{v}$, whose properties we have previously investigated, is for these reasons termed "the rotation * of the fluid."

The tensor $(1 + \mathbf{T}_1 dt)$ changes a volume in the ratio

$$1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dt = 1 + \nabla \cdot \mathbf{v}dt,$$

so that $\nabla \cdot \mathbf{v}dt$ represents the change per unit volume in time dt, and $\nabla \cdot \mathbf{v}$ the rate of change. If ρ is the density of the element of fluid considered at time t, and $\rho + d\rho$ its density at time $t + dt$, then the mass per unit volume must on the one hand equal ρ and on the other equal

$$(\rho + d\rho)(1 + \nabla \cdot \mathbf{v}dt),$$

i.e.

$$\rho = \rho + \rho\nabla \cdot \mathbf{v}dt + d\rho,$$

i.e. as we have already found

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\frac{d\rho}{dt}.$$

§ 15. Tensor Integrals

The conception of the tensor field which have been developed in a manner similar to that of the vector field and

* Many writers therefore use, in place of the vector $\nabla \times \mathbf{v}$ of an arbitrary field, the symbol rot $\mathbf{v}$. Maxwell writes curl $\mathbf{v}$ (the curling of the fluid).

field of vectorial area enables us to add to the three theorems on the transformation of the surface integrals

$$\int \mathbf{f}\,d\mathfrak{a}\,;\ \int \mathbf{p}\,d\mathfrak{a}\,;\ \int \mathbf{F}\,d\mathfrak{a}$$

into volume integrals, a fourth theorem by means of which the surface integral

$$\int \mathbf{T}\,d\mathfrak{a}$$

may be changed into a volume integral. In this $\mathbf{T}\,d\mathfrak{a}$ is a vector associated with the element of surface $d\mathfrak{a}$, and $d\mathfrak{a}$ is so chosen that the interior of the volume lies on the positive side of $d\mathfrak{a}$. If, for example, $\mathbf{T}$ is the tensor field of the stresses in an elastic body under any given boundary conditions, then $\mathbf{T}\,d\mathfrak{a}$ are the forces which must be applied to the elements of surface $d\mathfrak{a}$ on the boundary of any section in order to retain the equilibrium. The transformation into a volume integral occurs in the same way as with the three other cases, by supposing the volume cut up into a large number of small parallelepipeds, and the integration

$$\int \mathbf{T}\,d\mathfrak{a}$$

extended over the boundaries of all the portions. Every portion of the new surfaces which does not form part of the original surface then arises twice with opposite *senses* so that the two corresponding elements $\mathbf{T}\,d\mathfrak{a}$ annul each other, and therefore the total integral over the surfaces of all the portions equals the original surface integral.

The integration over a parallelepiped whose edges, proceeding from a corner determined by the position vector

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c},$$

are given by the vectors

$$\Delta x\mathbf{a}, \quad \Delta y\mathbf{b}, \quad \Delta z\mathbf{c}$$

(where $\mathbf{abc}$ is presumed a right-handed system), is effected once more by associating together pairs of parallel elements

of surface, for example, the two corresponding to the vectorial areas

$$\mathbf{bc}\Delta y\Delta z \text{ and } -\mathbf{bc}\Delta y\Delta z.$$

For the former $\mathbf{r}\,.\,\mathbf{a}^* = x$, and for the latter $\mathbf{r}\,.\,\mathbf{a}^* = x + \Delta x$. The two portions of the surface integral then provide to a first approximation

$$-\frac{\partial \mathbf{T}}{\partial x}\mathbf{bc}\Delta x\Delta y\Delta z.$$

By $\dfrac{\partial \mathbf{T}}{\partial x}$ is to be understood the tensor obtained when the tensor at the point x, y, z is subtracted from that at $x + \Delta x$, y, z, and on dividing by Δx the limit is approached; or what amounts to the same thing, the coefficients of the tensor are regarded as functions of x, y, z and differentiated partially with respect to x.

The three pairs of bounding surfaces then provide together

$$-\left(\frac{\partial \mathbf{T}}{\partial x}\mathbf{bc} + \frac{\partial \mathbf{T}}{\partial y}\mathbf{ca} + \frac{\partial \mathbf{T}}{\partial z}\mathbf{ab}\right)\Delta x\Delta y\Delta z,$$

or, as before, on treating the operator

$$|\,\nabla = \frac{\partial}{\partial x}\frac{\mathbf{bc}}{\mathbf{abc}} + \frac{\partial}{\partial y}\frac{\mathbf{ca}}{\mathbf{abc}} + \frac{\partial}{\partial z}\frac{\mathbf{ab}}{\mathbf{abc}}$$

as a vectorial area

$$-\mathbf{T}\,|\,\nabla\Delta x\Delta y\Delta z\,\mathbf{abc}$$

when Δx, Δy, Δz approach zero, the sum of all these portions give the volume integral

$$-\int \mathbf{T}\,|\,\nabla d\tau$$

where by $d\tau$, as before, is meant a right-handed volume element.

It follows that:

$$\int \mathbf{T}d\mathfrak{a} + \int \mathbf{T}\,|\,\nabla d\tau = 0.$$

If the volume element $d\tau$ of an elastic body are under the influence of forces

$$\mathbf{p}\,d\tau,$$

where $\mathbf{p}$ represents the corresponding vector field, and if by the application of certain surface forces,

$$\mathbf{T}\,d\mathfrak{G},$$

the element is maintained in equilibrium, then we must have

$$\int \mathbf{T}\,d\mathfrak{G} + \int \mathbf{p}\,d\tau = 0.$$

The same equation must, however, be valid when the two integrals are extended over the boundary and volume of an arbitrary portion of an elastic body. For if this portion is cut out, we must bring to bear on the boundary forces $\mathbf{T}\,d\mathfrak{G}$ in order to maintain equilibrium. Hence for every arbitrary portion of space

$$\int \mathbf{p}\,d\tau = \int \mathbf{T} \mid \nabla d\tau$$

and consequently

$$\mathbf{p} = \mathbf{T} \mid \nabla.$$

With the stress tensor, therefore, the vector

$$\mathbf{T} \mid \nabla$$

is simply the force, reckoned per unit of volume, acting at each point of the elastic body.

If the tensor $\mathbf{T}$ is thrown into the form

$$\mathbf{ea} + \mathbf{fb} + \mathbf{gc},$$

then

$$\mathbf{T} \mid \nabla = \mathbf{T} \cdot \nabla = \frac{\partial \mathbf{e}}{\partial x} + \frac{\partial \mathbf{f}}{\partial y} + \frac{\partial \mathbf{g}}{\partial z}.$$

Instead of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ let us insert a self-reciprocal right-handed system $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, then

$$\mathbf{p} = \mathbf{T} \mid \nabla$$

presents the conditions of equilibrium in the usual form. For then

$$\mathbf{e} = \mathbf{T}\mathbf{j}\mathbf{k}, \quad \mathbf{f} = \mathbf{T}\mathbf{k}\mathbf{i}, \quad \mathbf{g} = \mathbf{T}\mathbf{i}\mathbf{j}$$

are the stresses, measured per unit area, in the planes passing through the point under consideration and parallel to those of **jk, ki, ij.**

Using Kirchhoff's notation,

$$\begin{aligned}
\mathbf{e} &= X_x\mathbf{i} + Y_x\mathbf{j} + Z_x\mathbf{k} \\
\mathbf{f} &= X_y\mathbf{i} + Y_y\mathbf{j} + Z_y\mathbf{k} \\
\mathbf{g} &= X_z\mathbf{i} + Y_z\mathbf{j} + Z_z\mathbf{k}
\end{aligned}$$

and consequently

$$\begin{aligned}
\mathbf{T} \mid \nabla = &\left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right)\mathbf{i} \\
&+ \left(\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\right)\mathbf{j} \\
&+ \left(\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\right)\mathbf{k}.
\end{aligned}$$

If the force **p** acting on the unit volume is represented in the form

$$\mathbf{p} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k},$$

the condition of equilibrium

$$\mathbf{T} \mid \nabla = \mathbf{p}$$

assumes the usual form:

$$\begin{aligned}
\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} &= X \\
\frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} &= Y \\
\frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} &= Z.
\end{aligned}$$

The form
$$\mathbf{T} \mid \nabla$$

has the great advantage of being independent of the system of co-ordinates. It is merely necessary to insert the value

of ∇ in terms of co-ordinates in order to derive the conditions of equilibrium for any co-ordinate system. This applies with equal validity to curvilinear co-ordinates.

As we have already pointed out in Chap. II, § 11, in the latter case we have likewise

$$\nabla = \frac{\partial}{\partial \xi}\mathbf{e}^* + \frac{\partial}{\partial \eta}\mathbf{f}^* + \frac{\partial}{\partial \zeta}\mathbf{g}^*$$

and from this it follows that

$$\mathbf{T} \mid \nabla = \mathbf{T} \,.\, \nabla = \frac{\partial \mathbf{T}}{\partial \xi} \,.\, \mathbf{e}^* + \frac{\partial \mathbf{T}}{\partial \eta} \,.\, \mathbf{f}^* + \frac{\partial \mathbf{T}}{\partial \zeta} \,.\, \mathbf{g}^*.$$

Since, as was shown in Chap. II, § 11,

$$\frac{\partial \omega \mathbf{e}^*}{\partial \xi} + \frac{\partial \omega \mathbf{f}^*}{\partial \eta} + \frac{\partial \omega \mathbf{g}^*}{\partial \zeta} = 0,$$

we must also have

$$\mathbf{T} \,.\, \frac{\partial \omega \mathbf{e}^*}{\partial \xi} + \mathbf{T} \,.\, \frac{\partial \omega \mathbf{f}^*}{\partial \eta} + \mathbf{T} \,.\, \frac{\partial \omega \mathbf{g}^*}{\partial \zeta} = 0.$$

Hence for $\mathbf{T}/\nabla$ we may also write

$$\mathbf{T} \mid \nabla = \frac{1}{\omega}\frac{\partial \mathbf{T} \,.\, \omega \mathbf{e}^*}{\partial \xi} + \frac{1}{\omega}\frac{\partial \mathbf{T} \,.\, \omega \mathbf{f}^*}{\partial \eta} + \frac{1}{\omega}\frac{\partial \mathbf{T} \,.\, \omega \mathbf{g}^*}{\partial \zeta}.$$

The nine coefficients of the tensor are written for this purpose with double indices, and at the same time it is advisable to substitute for the letters ξ, η, ζ representing the variables, the symbols ξ_1, ξ_2, ξ_3, and similarly with the corresponding vectors e_1, e_2, e_3.

Let $\quad \mathbf{T} = \mathbf{T}^\lambda_\sigma \mathbf{e}^*_\sigma \mathbf{e}_\lambda$ [summed up for $\lambda = 1, 2, 3$
$\sigma = 1, 2, 3$]

then

$$\mathbf{T} \,.\, \omega \mathbf{e}^*_\lambda = \omega \mathbf{T}^\lambda_\sigma \mathbf{e}^*_\sigma \text{ [summed up for } \sigma = 1, 2, 3]$$

and consequently

$$\mathbf{T} \mid \nabla = \frac{1}{\omega}\frac{\partial \omega \mathbf{T}^\lambda_\sigma}{\partial \xi_\lambda}\mathbf{e}^*_\sigma + \mathbf{T}^\lambda_\sigma \frac{\partial \mathbf{e}^*_\sigma}{\partial \xi_\lambda} \text{ (sums } \lambda, \sigma).$$

For the scalar product:

$$-\frac{\partial \mathbf{e}^*_\sigma}{\partial \xi_\lambda} \cdot \mathbf{e}_\mu = \mathbf{e}^*_\sigma \cdot \frac{\partial \mathbf{e}_\mu}{\partial \xi_\lambda} = \mathbf{e}^*_\sigma \cdot \frac{\partial \mathbf{e}_\lambda}{\partial \xi_\mu} \dagger$$

Christoffel writes the symbol

$$\left\{ {\lambda\mu \atop \sigma} \right\},$$

so that we obtain

$$\frac{\partial \mathbf{e}^*_\sigma}{\partial \xi_\lambda} = \left(\frac{\partial \mathbf{e}^*_\sigma}{\partial \xi_\lambda} \cdot \mathbf{e}_\mu\right)\mathbf{e}^*_\mu \text{ [summed over } \mu = 1, 2, 3]$$
$$= -\left\{ {\lambda\mu \atop \sigma} \right\}\mathbf{e}^*_\mu$$

and

$$\mathbf{T} \mid \nabla = \left(\frac{1}{\omega}\sum_\lambda \frac{\partial \omega T^\lambda_\mu}{\partial \xi_\lambda} - \sum_{\lambda, \sigma} T^\lambda_\sigma \left\{ {\lambda\mu \atop \sigma} \right\}\right)\mathbf{e}^*_\mu$$

[summed over $\mu = 1, 2, 3$].

If $\mathbf{p}$ is the force vector, the conditions of equilibrium

$$\mathbf{T} \mid \nabla = \mathbf{p}$$

would be written in curvilinear co-ordinates in the form

$$\mathbf{p} \cdot \mathbf{e}_\mu = \sum_\lambda \frac{1}{\omega}\frac{\partial \omega T_\mu}{\partial \xi_\lambda} - \sum_{\lambda, \sigma} T^\lambda_\sigma \left\{ {\lambda\mu \atop \sigma} \right\}.$$

The terms $\mathbf{p} \cdot \mathbf{e}_\mu$ are the coefficients of the force referred to the system $\mathbf{e}^*_\mu$.

A few further remarks may be added concerning the derivation of the expressions:

$$\left\{ {\lambda\mu \atop \sigma} \right\} = \mathbf{e}^*_\sigma \cdot \frac{\partial \mathbf{e}_\mu}{\partial \xi_\lambda} = \mathbf{e}^*_\sigma \cdot \frac{\partial \mathbf{e}_\lambda}{\partial \xi}.$$

We have that:

$$\mathbf{e}^*_\sigma = \sum_\tau (\mathbf{e}^*_\sigma \cdot \mathbf{e}^*_\tau)\mathbf{e}_\tau.$$

† These equations follow from $\mathbf{e}^*_\sigma \cdot \mathbf{e}_\mu = 0$ or 1 and from $\frac{\partial \mathbf{e}_\mu}{\partial \xi_\lambda} = \frac{\partial^2 \mathbf{r}}{\partial \xi_\lambda \partial \xi_\mu}$.

Hence:

$$\left\{ {\lambda\mu \atop \sigma} \right\} = \sum_\tau (\mathbf{e}^*_\sigma \cdot \mathbf{e}^*_\tau)\left(\mathbf{e}_\tau \cdot \frac{\partial \mathbf{e}_\lambda}{\partial \xi_\mu}\right).$$

If the quantities $\mathbf{e}^*_\sigma \cdot \mathbf{e}^*_\tau$ are known then the quantities $\left\{ {\lambda\mu \atop \sigma} \right\}$ may be evaluated as soon as $\mathbf{e}_\tau \cdot \dfrac{\partial \mathbf{e}_\lambda}{\partial \xi_\mu}$ have been found. The latter, however, may be obtained from the quantities $\mathbf{e}_\tau \cdot \mathbf{e}_\lambda$. We have in fact:

$$\frac{\partial \mathbf{e}_\tau \cdot \mathbf{e}_\lambda}{\partial \xi_\mu} = \frac{\partial \mathbf{e}_\tau}{\partial \xi_\mu} \cdot \mathbf{e}_\lambda + \mathbf{e}_\tau \cdot \frac{\partial \mathbf{e}_\lambda}{\partial \xi_\mu}.$$

By cyclical substitution of τ, λ, μ the following additional equations are derived:

$$\frac{\partial \mathbf{e}_\lambda \cdot \mathbf{e}_\mu}{\partial \xi_\tau} = \frac{\partial \mathbf{e}_\lambda}{\partial \xi_\tau} \cdot \mathbf{e}_\mu + \mathbf{e}_\lambda \cdot \frac{\partial \mathbf{e}_\mu}{\partial \xi_\tau}.$$

$$\frac{\partial \mathbf{e}_\mu \cdot \mathbf{e}_\tau}{\partial \xi_\lambda} = \frac{\partial \mathbf{e}_\mu}{\partial \xi_\lambda} \cdot \mathbf{e}_\tau + \mathbf{e}_\mu \cdot \frac{\partial \mathbf{e}_\tau}{\partial \xi_\lambda}.$$

Of the six quantities on the right-hand sides certain pairs are equal; for example,

$$\frac{\partial \mathbf{e}_\tau}{\partial \xi_\mu} \cdot \mathbf{e}_\lambda = \mathbf{e}_\lambda \cdot \frac{\partial \mathbf{e}_\mu}{\partial \xi_\tau},$$

since

$$\frac{\partial \mathbf{e}_\tau}{\partial \xi_\mu} = \frac{\partial \mathbf{e}_\mu}{\partial \xi_\tau} = \frac{\partial^2 \mathbf{r}}{\partial \xi_\mu \partial \xi_\tau}.$$

Multiplying the third equation by -1 and adding the three equations together it follows that:

$$\frac{\partial \mathbf{e}_\tau \cdot \mathbf{e}_\lambda}{\partial \xi_\mu} + \frac{\partial \mathbf{e}_\lambda \cdot \mathbf{e}_\mu}{\partial \xi_\tau} - \frac{\partial \mathbf{e}_\mu \cdot \mathbf{e}_\tau}{\partial \xi_\lambda} = 2\frac{\partial \mathbf{e}_\tau}{\partial \xi_\mu} \cdot \mathbf{e}_\lambda.$$

The quantities

$$\frac{\partial \mathbf{e}_\tau}{\partial \xi_\mu} \cdot \mathbf{e}_\lambda = \frac{\partial \mathbf{e}_\mu}{\partial \xi_\tau} \cdot \mathbf{e}_\lambda$$

are represented by Christoffel by:

$$\left[{\mu\tau \atop \lambda} \right].$$

The quantities $\begin{bmatrix}\mu\tau\\ \lambda\end{bmatrix}$ may be derived from the coefficients $\mathbf{e}_\lambda \cdot \mathbf{e}_\mu$ of the expression:

$$d\mathbf{r} \cdot d\mathbf{r} = \sum(\mathbf{e}_\lambda \cdot \mathbf{e}_\mu) d\xi_\lambda d\xi_\mu.$$

The terms $\mathbf{e}^*_\lambda \cdot \mathbf{e}^*_\mu$ follow on reversing the equations

$$\mathbf{e}_\lambda = \sum_\mu (\mathbf{e}_\mu \cdot \mathbf{e}_\lambda)\mathbf{e}^*_\mu$$

or else from

$$\mathbf{e}^*_\lambda = \nabla\xi^\lambda.$$

From the quantities $\begin{bmatrix}\mu\tau\\ \lambda\end{bmatrix}$ and $\mathbf{e}^*_\lambda \cdot \mathbf{e}^*_\mu$, the quantities $\begin{Bmatrix}\lambda\mu\\ \sigma\end{Bmatrix}$ are then found.

§ 16. Cogradience and Contragradience

If a vector $\mathbf{p}$ is numerically derived from three vectors $\mathbf{e}, \mathbf{f}, \mathbf{g}$,

$$\mathbf{p} = x\mathbf{e} + y\mathbf{f} + z\mathbf{g}$$

and if instead of $\mathbf{e}, \mathbf{f}, \mathbf{g}$ three other unit vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ ar introduced:

$$\text{I.} \qquad \begin{aligned} \mathbf{e} &= a_1\mathbf{a} + b_1\mathbf{b} + c_1\mathbf{c} \\ \mathbf{f} &= a_2\mathbf{a} + b_2\mathbf{b} + c_2\mathbf{c} \\ \mathbf{g} &= a_3\mathbf{a} + b_3\mathbf{b} + c_3\mathbf{c} \end{aligned}$$

then we obtain

$$\mathbf{p} = \xi\mathbf{a} + \eta\mathbf{b} + \zeta\mathbf{c},$$

where

$$\text{II.} \qquad \begin{aligned} \xi &= a_1x + a_2y + a_3z \\ \eta &= b_1x + b_2y + b_3z \\ \zeta &= c_1x + c_2y + c_3z. \end{aligned}$$

This indicates that the new coefficients ξ, η, ζ do not stand in the same relation to the old ones, x, y, z, as the new unit vectors a, b, c to the old e, f, g. If, on the other hand, the reciprocal unit vectors $\mathbf{e}^*, \mathbf{f}^*, \mathbf{g}^*$ and $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ those reciprocal to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be considered, then it is apparent that $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ are derived from $\mathbf{e}^*, \mathbf{f}^*, \mathbf{g}^*$ by means of exactly the same equations as when ξ, η, ζ is derived from x, y, z.

For the tensor **T** which leaves every vector unaltered, as has already been indicated, can be thrown into the form :

$$\mathbf{T} = \mathbf{ee}^* + \mathbf{ff}^* + \mathbf{gg}^* = \mathbf{aa}^* + \mathbf{bb}^* + \mathbf{cc}^*.$$

Inserting the expressions I. for **e**, **f**, **g** and arranging the terms according to **a**, **b**, **c** we find

$$\begin{aligned} \mathbf{T} = {} & \mathbf{a}(a_1\mathbf{e}^* + a_2\mathbf{f}^* + a_3\mathbf{g}) \\ & + \mathbf{b}(b_1\mathbf{e}^* + b_2\mathbf{f}^* + b_3\mathbf{g}^*) \\ & + \mathbf{c}(c_1\mathbf{e}^* + c_2\mathbf{f}^* + c_3\mathbf{g}^*), \end{aligned}$$

and consequently

$$\begin{aligned} \mathbf{a}^* &= a_1\mathbf{e}^* + a_2\mathbf{f}^* + a_3\mathbf{g}^* \\ \mathbf{b}^* &= b_1\mathbf{e}^* + b_2\mathbf{f}^* + b_3\mathbf{g}^* \\ \mathbf{c}^* &= c_1\mathbf{e}^* + c_2\mathbf{f}^* + c_3\mathbf{g}^*, \end{aligned}$$

from which it appears that the new reciprocal unit vectors $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ stand in the same relation to the old reciprocal unit vectors $\mathbf{e}^*$, $\mathbf{f}^*$, $\mathbf{g}^*$ as the new coefficients ξ, η, ζ stand to the old x, y, z.

If the vector **p** is derived from the reciprocal unit vectors

$$\mathbf{p} = x^*\mathbf{e}^* + y^*\mathbf{f}^* + z^*\mathbf{g}^* = \xi^*\mathbf{a}^* + \eta^*\mathbf{b}^* + \zeta^*\mathbf{c}^*,$$

then the coefficients ξ^*, η^*, ζ^* bear to x^*, y^*, z^* the same relation as the unit vectors **a**, **b**, **c** bear to **e**, **f**, **g**. For if for $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ the expressions in terms of $\mathbf{e}^*$, $\mathbf{f}^*$, $\mathbf{g}^*$ are inserted and the terms arranged according to $\mathbf{e}^*$, $\mathbf{f}^*$, $\mathbf{g}^*$, then on comparing with $x^*\mathbf{e}^* + y^*\mathbf{f}^* + z^*\mathbf{g}^*$ we obtain the following equations :

$$\begin{aligned} x^* &= a_1\xi^* + b_1\eta^* + c_1\zeta^* \\ y^* &= a_2\xi^* + b_2\eta^* + c_2\zeta^* \\ z^* &= a_3\xi^* + b_3\eta^* + c_3\zeta^*, \end{aligned}$$

which correspond exactly to equations I.

In order to express this property between the coefficients and the unit vectors shortly, we say, "the coefficients are transformed contragradiently to the unit vectors to which they relate and cogradiently to the reciprocal unit vectors." The word cogradient implies the same type of transforma-

tion in the passage to new unit vectors and the corresponding new coefficients.

Using affixes to distinguish the various unit vectors and coefficients, then the contragradience is best indicated by using a suffix in one case and an index in the other. Accordingly we write

$$\mathbf{p} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 = x_1\mathbf{e}^1 + x_2\mathbf{e}^2 + x_3\mathbf{e}^3,$$

indicating by the position of the index that for instance $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, are contragradient to $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$, while x^1, x^2, x^3 are cogradient to x_1, x_2, x_3.

At the same time, for the sake of brevity, it is advisable to avoid writing the sum out in full but merely

$$\mathbf{p} = x^i\mathbf{e}_i = x_i\mathbf{e}^i$$

where it is implied by the occurrence of the affix i twice, that the sum is intended to be taken over $i = 1, 2, 3$. In analogous manner we may deal with the coefficients and units of tensors. We write

$$\mathbf{T} = a^{ik}\mathbf{e}_i\mathbf{e}_k,$$

where the indices run through the values i, k independently and the sum is taken of the nine terms. The nine quantities a^{ik} are the coefficients of the tensor and the nine tensors $\mathbf{e}_i\mathbf{e}_k$ are the units to which the coefficients refer. In the passage to the new unit vectors $\mathbf{f}_i$, there enter in place of the nine unit tensors $\mathbf{e}_i\mathbf{e}_k$, the new unit tensors $\mathbf{f}_\lambda\mathbf{f}_\mu$ linearly connected with them,

$$\mathbf{T} = a^{ik}\mathbf{e}_i\mathbf{e}_k = b^{\lambda\mu}\mathbf{f}_\lambda\mathbf{f}_\mu.$$

The new coefficients $b^{\lambda\mu}$ are then likewise linearly expressed in terms of the old coefficients.

Let $$\mathbf{e}_i = f_i^\lambda\mathbf{f}_\lambda$$

and therefore $$\mathbf{f}^\lambda = f_i^\lambda\mathbf{e}^i,$$

where again the summation is to be extended on the right-hand side over the various values of the index appearing above and below. Then

$$\mathbf{e}_i\mathbf{e}_k = f_i^\lambda f_k^\mu\,\mathbf{f}_\lambda\mathbf{f}_\mu$$

and $$\mathbf{f}^\lambda \mathbf{f}^\mu = f^\lambda_i f^\mu_k \mathbf{e}^i \mathbf{e}^k$$

and $$\mathbf{T} = a^{ik} f^\lambda_i f^\mu_k \mathbf{f}_\lambda \mathbf{f}_\mu = b^{\lambda\mu} \mathbf{f}_\lambda \mathbf{f}_\mu,$$

consequently

$$b^{\lambda\mu} = f^\lambda_i f^\mu_k a^{ik}.$$

This implies that the a^{ik} are transformed in the passage to the new unit vectors contragradiently to $\mathbf{e}_i\mathbf{e}_k$ and cogradiently to $\mathbf{e}^i\mathbf{e}^k$. This property of the a^{ik} is expressed by putting both indices above. If the tensor $\mathbf{T}$ is taken with reference to the reciprocal units $\mathbf{e}^i\mathbf{e}^k$, the corresponding coefficients for a similar reason are written with suffixes:

$$\mathbf{T} = a^{ik}\mathbf{e}_i\mathbf{e}_k = a_{ik}\mathbf{e}^i\mathbf{e}^k.$$

Use might also be made of $\mathbf{e}_i$ in place of the first vector and $\mathbf{e}^k$ for the second, or vice versa, and then the following forms are derived for the same tensor:

$$\mathbf{T} = a^i_k \mathbf{e}_i \mathbf{e}^k,$$

or

$$\mathbf{T} = a^i_k \mathbf{e}^k \mathbf{e}_i.$$

These are termed "mixed" forms.

If the tensor $\mathbf{T}$ is "self-conjugate or symmetric," then, as explained above, it is unaltered by interchanging the two vectors in each of its terms, i.e.

$$a^{ik} = a^{ki};\ a_{ik} = a_{ki};\ a^i_k = \bar{a}^i_k.$$

A symmetric tensor then possesses merely one mixed form; and conversely if a tensor possesses only one mixed form it is symmetric.

For if in the form

$$\mathbf{T} = a^i_k \mathbf{e}_i \mathbf{e}^k$$

the two vectors in each term can be interchanged without altering the tensor, it is self-conjugate.

An asymmetric tensor is transformed into its opposite if the two vectors are interchanged in each term. For an asymmetric tensor

$$a^{ik} = -a^{ki};\quad a_{ik} = -a_{ki};$$

and at the same time

$$a^{ii} = 0; \quad a_{ii} = 0.$$

As we have seen above, the vectorial multiplication of a vector $\mathbf{p}$ with a second vector may be replaced by the application of an asymmetric tensor $\mathbf{T}$ to $\mathbf{p}$.

Then, if

$$\mathbf{T} = a^{ik}\mathbf{e}_i\mathbf{e}_k \text{ and } \mathbf{p} = p_\lambda \mathbf{e}^\lambda$$

we get

$$\mathbf{T} \cdot \mathbf{p} = a^{ik}\mathbf{e}_i(\mathbf{e}_k \cdot \mathbf{p}) = a^{ik}p_k\mathbf{e}_i.$$

Hence the coefficients of the vector $\mathbf{T} \cdot \mathbf{p}$ are equal to the three sums $a^{ik}p_k$, or written out in full:

$$\begin{gathered} a^{11}p_1 + a^{12}p_2 + a^{13}p_3, \\ a^{21}p_1 + a^{22}p_2 + a^{23}p_3, \\ a^{31}p_1 + a^{32}p_2 + a^{33}p_3, \end{gathered}$$

or, remembering that

$$a^{11} = a^{22} = a^{33} = 0, \text{ and } a^{ik} = -a^{ki},$$

the three coefficients are:

$$a^{12}p_2 - a^{31}p_3, \quad -a^{12}p_1 + a^{23}p_3, \quad a^{31}p_1 - a^{23}p_2.$$

These are the coefficients of the vectorial product

$$\mathbf{p} \times \mathbf{q},$$

where the vector $\mathbf{q}$ equals the representation of the vectorial area

$$\mathbf{Q} = a^{23}\mathbf{e}_2\mathbf{e}_3 + a^{31}\mathbf{e}_3\mathbf{e}_1 + a^{12}\mathbf{e}_1\mathbf{e}_2.$$

In place of the vector, its representation may be introduced, viz. the vectorial area:

$$\mathbf{P} = \frac{p_1\mathbf{e}_2\mathbf{e}_3}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3} + \frac{p_2\mathbf{e}_3\mathbf{e}_1}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3} + \frac{p_3\mathbf{e}_1\mathbf{e}_2}{\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3},$$

and instead of

$$\mathbf{T} \cdot \mathbf{p} = \mathbf{p} \times \mathbf{q}$$

we would write

$$\mathbf{TP} = \mathbf{PQ}.$$

Every vectorial area $\mathbf{Q}$ stands accordingly in a close relation to an asymmetric tensor $\mathbf{T}$, and conversely. The vectorial area can be derived numerically from the three external products $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2$ by means of the three corresponding coefficients a^{23}, a^{31}, a^{12}:

$$\mathbf{Q} = a^{23}\mathbf{e}_2\mathbf{e}_3 + a^{31}\mathbf{e}_3\mathbf{e}_1 + a^{12}\mathbf{e}_1\mathbf{e}_2.$$

Writing a^{32}, a^{13}, a^{21} for $-a^{23}$, $-a^{31}$, $-a^{12}$ we may equally well put

$$\mathbf{Q} = a^{32}\mathbf{e}_3\mathbf{e}_2 + a^{13}\mathbf{e}_1\mathbf{e}_3 + a^{21}\mathbf{e}_2\mathbf{e}_1,$$

since the external products $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2$ are equal and opposite to $\mathbf{e}_3\mathbf{e}_2$, $\mathbf{e}_1\mathbf{e}_3$, $\mathbf{e}_2\mathbf{e}_1$.

Adding both expressions, however, we find

$$2\mathbf{Q} = a^{ik}\mathbf{e}_i\mathbf{e}_k,$$

the sum on the right-hand side being extended over the values $i = 1, 2, 3$ and $k = 1, 2, 3$.

The asymmetric tensor $\mathbf{T}$ has exactly the same form,

$$\mathbf{T} = a^{ik}\mathbf{e}_i\mathbf{e}_k,$$

except that the six quantities do not here mean external products, but the six unit tensors from which the asymmetric tensor $\mathbf{T}$ is numerically derived.

In this case $\mathbf{e}_i\mathbf{e}_k - \mathbf{e}_k\mathbf{e}_i$, but the six tensors $\mathbf{e}_i\mathbf{e}_k$ are merely numerically independent of each other. As has already been remarked in § 9, from the asymmetric tensor we derive the vectorial area corresponding to it, by replacing the unit tensors $\mathbf{e}_i\mathbf{e}_k$ by the external products $\mathbf{e}_i\mathbf{e}_k$ and dividing by two. In the expression for $\mathbf{T}$ the terms may be associated in pairs of the type $\mathbf{e}_i\mathbf{e}_k$ and $\mathbf{e}_k\mathbf{e}_i$:

$$\mathbf{T} = a^{23}(\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_2) + a^{31}(\mathbf{e}_3\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_3) + a^{12}(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1).$$

In this form $\mathbf{T}$ is derived numerically from the three asymmetric tensors:

$$\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_3\mathbf{e}_2;\quad \mathbf{e}_3\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_3;\quad \mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1;$$

that is to say, no more than three linearly independent asymmetric unit tensors are required. The coefficients of **T** in relation to these three asymmetric tensors coincide with the coefficients of the vectorial area **Q** in relation to the three unit vectorial areas $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2$. Written in contragradient units, we have

$$\mathbf{T} = a_{ik}\mathbf{e}^i\mathbf{e}^k$$

and

$$\mathbf{Q} = \tfrac{1}{2}a_{ik}\mathbf{e}^i\mathbf{e}^k,$$

where again in the expression for **Q** each term $\mathbf{e}^i\mathbf{e}^k$ represents the external product of the two vectors. Accordingly both the vectorial product of a vector **p** with a vector **q**, which denotes the representation of **q**, and the external product of a vectorial area **P** with a vectorial area **Q** may be expressed by means of the asymmetric tensor **T** belonging to the vectorial area **Q**. For, as we saw before:

$$\mathbf{T}\,.\,\mathbf{p} = \mathbf{p}\,.\,\mathbf{q} \text{ and } \mathbf{TP} = \mathbf{PQ}.$$

In certain circumstances there is a distinct advantage in utilising this and replacing the vectorial product or the external product of two vectorial areas by an asymmetric tensor.

The velocity **v**, for example, of a point of a rigid body rotating about an axis may be presented in the form

$$\mathbf{v} = \mathbf{T}\,.\,\mathbf{r}.$$

In this **r** is the position vector from some point of the axis to the point of the body under consideration. **T** is an asymmetric tensor,

$$\mathbf{T} = a^{ik}\mathbf{e}_i\mathbf{e}_k(a^{ik} = -a^{ki})$$

and the vectorial area

$$\mathbf{Q} = \tfrac{1}{2}a^{ik}\mathbf{e}_i\mathbf{e}_k,$$

obtained from **T** by regarding $\mathbf{e}_i\mathbf{e}_k$ as an external product, is at right angles to the axis of rotation, and its numerical value is the velocity of rotation. The direction of rotation is opposed to that of the *sense* of **Q**.

As an illustration let us consider the motion of a rigid

body about a fixed point, there being no forces in operation other than those implied by this condition. Let $\mathbf{v}$ be the velocity of the element of mass dm and $\mathbf{r}$ its position vector measured from the fixed point. Then the sum of the angular momenta

$$\int \mathbf{rv}\,dm$$

is a constant vectorial area which we will represent by $\mathbf{F}$. For the vector $\mathbf{v}$ we write

$$\mathbf{v} = \mathbf{T}\,.\,\mathbf{r} = a^{ik}x_k\mathbf{e}_i,$$

so that we have

$$\mathbf{F} = a^{ik}t_k^\lambda\mathbf{e}_\lambda\mathbf{e}_i,$$

where

$$t_k^\lambda = \int x^\lambda x_k\,dm,$$

$$\mathbf{r} = x^\lambda\mathbf{e}_\lambda = x_k\mathbf{e}^k.$$

The quantities t_k^λ are the coefficients of a symmetric tensor:

$$\mathbf{T}' = t_k^\lambda\mathbf{e}_\lambda\mathbf{e}^k = \int \mathbf{rr}\,dm.$$

Accordingly the unit vectors may be so chosen that they coincide with the principal axes of $\mathbf{T}'$ and constitute a self-reciprocal right-handed system. This system is fixed to the rigid body as must follow directly from the definition of $\mathbf{T}'$:

$$\mathbf{T}' = \int \mathbf{rr}\,dm.$$

For the position vector $\mathbf{r}$ of each element dm is fixed with reference to the body. $\mathbf{T}'$ may be termed the inertia tensor of the body. Its principal axes are the principal axes of inertia for the fixed point of the body.

For this self-reciprocal system there is now no longer any necessity to distinguish between the upper and the lower affixes, while, moreover, the coefficients t_k^λ when k is not equal to λ vanish.

The vectorial area $\mathbf{F}$ which represents the total angular momentum may then be expressed in the form

$$\mathbf{F} = a^{ik}t^k_k\mathbf{e}_k\mathbf{e}_i,$$

or writing it in full, and remembering that $a^{ik} = -a^{ki}$,

$$\mathbf{F} = a^{32}(t^3_3 + t^2_2)\mathbf{e}_2\mathbf{e}_3 + a^{13}(t^1_1 + t^3_3)\mathbf{e}_3\mathbf{e}_1 + a^{21}(t^2_2 + t^1_1)\mathbf{e}_1\mathbf{e}_2.$$

The differential equations are completely expressed by the condition

$$\frac{d\mathbf{F}}{dt} = 0,$$

i.e. by the condition that the angular momentum does not vary. To set these equations out in terms of the coefficients it must be noted that the unit vectors $\mathbf{e}_1$ vary according to the formula

$$\frac{d\mathbf{e}_i}{dt} = \mathbf{T} \cdot \mathbf{e}_i = a^{ki}\mathbf{e}_k.$$

For if $\mathbf{e}_i$ is drawn from the fixed point it is the position vector of a point fixed in the body, the velocity of which is provided by the rotation tensor $\mathbf{T}$. The quantities

$$t^3_3 + t^2_2,\ t^1_1 + t^3_3,\ t^2_2 + t^1_1$$

are the three principal moments of inertia, and for these we will use the letters A_1, A_2, A_3.

On differentiating the first term of $\mathbf{F}$ we find:

$$A_1\frac{da^{32}}{dt}\mathbf{e}_2\mathbf{e}_3 + A_1a^{32}a^{12}\mathbf{e}_1\mathbf{e}_3 + A_1a^{32}a^{13}\mathbf{e}_2\mathbf{e}_1.$$

The differential coefficients of the other two terms are derived by cyclic substitution. The coefficient corresponding to $\mathbf{e}_2\mathbf{e}_3$ in $\frac{d\mathbf{F}}{dt}$ is then equal to:

$$A_1\frac{da^{32}}{dt} + (A_3 - A_2)a^{13}a^{21}.$$

Now since the vanishing of $\frac{d\mathbf{F}}{dt}$ requires the vanishing of its three coefficients we obtain the equation

$$A_1\frac{da^{32}}{dt} + (A_3 - A_2)a^{13}a^{21} = 0$$

and other two derived from this by cyclic substitution of the indices.

As proved above, the direction of rotation is the reverse of that of the sense of the vectorial area:

$$\mathbf{Q} = a^{23}\mathbf{e}_2\mathbf{e}_3 + a^{31}\mathbf{e}_3\mathbf{e}_1 + a^{12}\mathbf{e}_1\mathbf{e}_2.$$

If it is desired, as is usually the case, to introduce the vector $\mathbf{q}$ which is parallel to the axis of rotation and represents the angular velocity in the usual manner in the sense of a right-handed screw, then we must set:

$$\mathbf{q} = - \mid \mathbf{Q} = a^{32}\mathbf{e}_1 + a^{13}\mathbf{e}_2 + a^{21}\mathbf{e}_3.$$

These differential equations for the coefficients of the rotational velocity were first set up by Euler. To effect their integration and deduce the circumstances of the motion, it is better to return to the vectorial areas $\mathbf{F}$ and $\mathbf{Q}$.

Concerning $\mathbf{F}$, we know that it is constant, and consequently the square of its numerical value is a constant scalar quantity:

$$\mathbf{F} \mid \mathbf{F} = c_1.$$

The vectorial area $\mathbf{Q}$ provides with $\mathbf{F}$ the scalar:

$$\mathbf{Q} \mid \mathbf{F} = (a^{32})^2 A_1 + (a^{13})^2 A_2 + (a^{21})^2 A_3.$$

Since the coefficients of $\mathbf{Q}$ represent the rotational velocities about the three axes, and the quantities A are the moments of inertia, $\mathbf{Q} \mid \mathbf{F}$ is twice the kinetic energy and thus a second scalar constant is at once derived:

$$\mathbf{Q} \mid \mathbf{F} = c_2.$$

Both constants might have been derived as constants of integration from the differential equations. We need merely

multiply them in order by a^{32}, a^{13}, a^{21} and with A_1a^{32}, A_2a^{13}, A_3a^{21} and add, so that we get the equations

$$A_1a^{32}\frac{da^{32}}{dt} + A_2a^{13}\frac{da^{13}}{dt} + A_3a^{21}\frac{da^{21}}{dt} = 0,$$

and

$$A_1^2a^{32}\frac{da^{32}}{dt} + A_2^2a^{13}\frac{da^{13}}{dt} + A_3^2a^{21}\frac{da^{21}}{dt} = 0$$

and the integrals of these provide us with the equations

$$\mathbf{F} \mid \mathbf{F} = c_1 \text{ and } \mathbf{Q} \mid \mathbf{F} = c_2.$$

These two equations enable the three components of the angular velocity to be expressed algebraically in terms of one of them, which itself is obtained as an elliptic function * of t. It is not, however, for this reason that these equations have been developed here but rather because they are eminently suitable for providing a clear representation of the motion.

By means of the symmetric tensor

$$\mathbf{T} = A_1\mathbf{e}_1\mathbf{e}_1 + A_2\mathbf{e}_2\mathbf{e}_2 + A_3\mathbf{e}_3\mathbf{e}_3$$

the representations of $-\mathbf{F}$ and $-\mathbf{Q}$, viz.

$$\mathbf{f} = -\mid\mathbf{F} \text{ and } \mathbf{q} = -\mid\mathbf{Q},$$

are directly connected.

$$\mathbf{f} = \mathbf{T}\,.\,\mathbf{q},$$

or as we may also express it, $\mathbf{f}$ is one half the gradient of the scalar function $(\mathbf{T}\,.\,\mathbf{q})\,.\,\mathbf{q}$, belonging to the symmetric tensor (cf. Chap. III, § 5). The scalar function equals

$$(\mathbf{T}\,.\,\mathbf{q})\,.\,\mathbf{q} = \mathbf{f}\,.\,\mathbf{q} = \mathbf{F} \mid \mathbf{Q} = A_1x_1^2 + A_2x_2^2 + A_3x_3^2,$$

where the coefficients of $\mathbf{q}$ are represented by x_1, x_2, x_3.

When equated to a constant it provides the equation of the momental ellipsoid. The vector $\mathbf{q}$, which represents the instantaneous axis of rotation and the angular velocity

* Compare, for example, G. Kirchhoff: Mechanik, "7th lecture," § 1.

may accordingly be imagined as a radius leading from a fixed point to a point of the ellipsoid:

$$A_1x_1^2 + A_2x_2^2 + A_3x_3^2 = c_2.$$

The vector of angular momentum $\mathbf{f}$ as half the gradient of the scalar function must have the direction of the perpendicular that may be dropped from the fixed point on to the tangent plane at the end of $\mathbf{q}$. Accordingly this perpendicular is equal to

$$\frac{\mathbf{f}\cdot\mathbf{q}}{\mathbf{f}\cdot\mathbf{f}}\mathbf{f} = \frac{c_2}{c_1}\mathbf{f}.$$

For it has the direction of $\mathbf{f}$, and a length $\dfrac{\mathbf{f}\cdot\mathbf{q}}{\sqrt{\mathbf{f}\cdot\mathbf{f}}}$. Since, however, both $\mathbf{f}$ and $\mathbf{f}\cdot\mathbf{q}$ are constant, it follows that during the motion of the rigid body this perpendicular must always maintain the same direction and length, i.e. the tangent plane must always remain fixed. In other words, the ellipsoid can only occupy such positions as allows it to touch the fixed tangent plane. The vector joining the fixed centre of the ellipsoid to the point of contact gives the instantaneous axis of rotation and the angular velocity. The point of contact at each instant lies on the axis of rotation and therefore has zero velocity. Imagine the succession of points of contact drawn both on the tangent plane and on the surface of the ellipsoid; they determine two curves one of which is fixed and the other rolls upon it. The possible curves on the ellipsoid are algebraic for they are lines of intersection of the ellipsoid

$$\mathbf{f}\cdot\mathbf{q} = A_1x_1^2 + A_2x_2^2 + A_3x_3^2 = c_2$$

with the ellipsoid

$$\mathbf{f}\cdot\mathbf{f} = A_1^2x_1^2 + A_2^2x_2^2 + A_3^2x_3^2 = c_1.$$

To construct a picture of the sequence of lines of intersection suppose the former ellipsoid held fixed. The squares of its semi-axes are:

$$c_2/A_1,\ c_2/A_2,\ c_2/A_3.$$

The ellipsoid $\mathbf{f}\,.\,\mathbf{f} = c_1$, on the other hand, can alter by allowing c_1 to increase. The squares of its semi-axes are:

$$c_1/A_1^2,\ c_1/A_2^2,\ c_1A_3^2.$$

Let $A_1 < A_2 < A_3$ then in both ellipsoids the first semi-axis is the greatest, and the last is the smallest. In the ellipsoid $\mathbf{f}\,.\,\mathbf{f} = c_1$, however, the relative differences of the three axes are greater, i.e. the ratio of the largest to the middle one or to the smallest is greater than the corresponding ratio for the other ellipsoid. The smallest value of c_1 for which the two ellipsoids have points in common will therefore be that for which the major axes coincide:

$$c_1/A_1^2 = c_2/A_1 \text{ or } c_1 = c_2A_1;$$

the largest value of c_1 for which they still have points in common is that for which the smallest axes coincide:

$$c_1/A_3^2 = c_2/A_3 \text{ or } c_1 = c_2A_3.$$

If c_1 increases slightly beyond the smallest value the line of intersection consists of two closed curves which encircle the end-point of the major axis of the momental ellipsoid, while if c_1 is slightly smaller than the largest value the line of intersection is composed of two closed curves encircling the small axis. If c_1 increases from the smallest value c_2A_1 to the largest c_2A_3 the closed curves of the first type must transform into those of the second type. The transition is determined by the curves of intersection obtained from $c_1 = c_2A_2$, when the two ellipsoids touch at the terminal points of the middle axis. The curve of intersection has two double points at these points. It divides the surface of the momental ellipsoid into four separate portions. Two portions which surround the smallest axis lie outside the ellipse $\mathbf{f}\,.\,\mathbf{f} = c_2A_2$ and two portions surrounding the largest axis lie inside. For $c_1 < c_2A_2$ the curve of intersection encircles the greatest axis, for $c_1 > c_2A_2$ the smallest.

For the rotation about the principal axes

$$\mathbf{q} = \pm\sqrt{\frac{c_2}{A_1}}\mathbf{e}_1, \text{ or } \mathbf{q} = \pm\sqrt{\frac{c_2}{A_2}}\mathbf{e}_2, \text{ or } \mathbf{q} = \pm\sqrt{\frac{c_2}{A_2}}\mathbf{e}_3$$

and corresponding to each of these $\mathbf{f} = A_1\mathbf{q}$, or $\mathbf{f} = A_2\mathbf{q}$, or $\mathbf{f} = A_3\mathbf{q}$. In all three cases the equations of motion are satisfied if the rotational vector $\mathbf{q}$ is assumed constant. There is, however, an essential difference between rotation around the middle axis and the rotations around the other two axes. A small change of the angular momentum $\mathbf{f}$, for instance, communicated by a small impulse would throw the rotation vector $\mathbf{q}$ slightly out of the principal axis. With the greatest and least axes the end-point will describe a closed curve on the momental ellipsoid about the terminal point of the corresponding axis, so that it always remains in the vicinity of that axis. And since, moreover, the normals of the momental ellipsoid in the neighbourhood of the axis make only a small angle with it, it follows that during the motion the axis will be unable during the motion of the rigid body to depart to any great extent from the constant momentum vector $\mathbf{f}$. With the middle axis, on the other hand, by a slight impulse the end-point of $\mathbf{q}$ will be thrown on to a curve along which it will move until it reaches the vicinity of the opposite end of the axis, so that the rotation vector in the intermediate range will make all angles with the middle axis.

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